# Solution to Maxwell's field equations 

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#### Abstract

One of the problems Classical Electrodynamics tries to solve is that to integrate the system of Maxwell's equations (1865), that is to obtain an analytical expression which would allow to calculate the electrical and magnetic fields given the distributions of charge and current.


## NOTATION

In the next pages, we will use the following notation:

- $\boldsymbol{\nabla} \equiv\left(\partial_{x}, \partial_{y}, \partial_{z}\right)$, where $\partial_{i}$ indicates the partial derivative with respect of the i-th component;
- Given a scalar field, its gradient will be indicated as $\boldsymbol{\nabla} \varphi \equiv \operatorname{grad} \varphi$;
- Given a vector field $\mathbf{F}$, we will write its divergence and curl respectively $\boldsymbol{\nabla} \cdot \mathbf{F} \equiv \operatorname{div} \mathbf{F}$ and $\nabla \wedge \mathbf{F} \equiv \operatorname{curl} \mathbf{F} ;$
- The Laplace operator will be indicated as $\nabla^{2}$;
- Given a set $\Omega$, its boundary will be indicated as $\partial \Omega$.

In the next pages, we will use the so called indicial notation, in order to facilitate the execution of particular calculations, and the Einstein summation convention (which is to sum when an index variable appears twice in a single term over all possible values of the index). By virtue of this notation, any vector ( $\left\{\mathbf{e}_{i}\right\}$ is an orthonormal basis, in particular $\left.\mathbf{e}_{i} \cdot \mathbf{e}_{j}=\delta_{i j}\right)$ :

$$
\begin{equation*}
\mathbf{a}=\sum_{i} a_{i} \mathbf{e}_{i} \tag{.1}
\end{equation*}
$$

will be written in Einstein notation as

$$
\begin{equation*}
\mathbf{a}=a_{i} \mathbf{e}_{i} \tag{.2}
\end{equation*}
$$

The inner product among two vectors will be written as

$$
\begin{align*}
\mathbf{a} \cdot \mathbf{b} & =\left(a_{i} \mathbf{e}_{i}\right) \cdot\left(b_{j} \mathbf{e}_{j}\right)= \\
& =a_{i} b_{j} \mathbf{e}_{i} \cdot \mathbf{e}_{j}= \\
& =a_{i} b_{j} \delta_{i j}= \\
& =a_{i} b_{i} \tag{.3}
\end{align*}
$$

[^0]while the cross product will be written as
\[

$$
\begin{equation*}
\mathbf{a} \wedge \mathbf{b}=\epsilon_{i j k} a_{j} b_{k} \mathbf{e}_{i} \tag{.4}
\end{equation*}
$$

\]

where $\epsilon_{i j k}$ is the Levi-Civita symbol, defined by
$\epsilon_{i j k}= \begin{cases}1 & \text { if }(i, j, k)=\{(1,2,3),(2,3,1),(3,1,2)\} \\ 0 & \text { if } i=j \text { or } j=k \text { or } k=i \\ -1 & \text { if }(i, j, k)=\{(1,3,2),(3,2,1),(2,1,3)\}\end{cases}$
For what concerns differential operators, we have

- Gradient: $\nabla \varphi=\partial_{i} \varphi \mathbf{e}_{i} ;$
- Divergence: $\boldsymbol{\nabla} \cdot \mathbf{F}=\partial_{i} F_{i}$;
- Curl: $\boldsymbol{\nabla} \times \mathbf{F}=\epsilon_{i j k} \partial_{j} F_{k} \mathbf{e}_{i}$;
- Laplacian: $\nabla^{2} \varphi=\partial_{i} \partial_{i} \varphi$.


## I. PRELIMINARY DEFINITIONS

Theorem 1 (Gauss-Green).
Suppose $\Omega$ is a subset of $\mathbb{R}^{n}$ which is compact and has a piecewise smooth boundary $S=\partial \Omega$. If $\mathbf{F}$ : $\Omega \rightarrow \mathbb{R}^{n}$ is a continuously differentiable vector field defined on a neighborhood of $\Omega$, then we have

$$
\begin{equation*}
\int_{\Omega}(\boldsymbol{\nabla} \cdot \mathbf{F}) \mathrm{d} V=\oint_{\partial V}(\mathbf{F} \cdot \mathbf{n}) \mathrm{d} S \tag{I.1}
\end{equation*}
$$

Theorem 2 (Curl).
Suppose $\Omega$ is a subset of $\mathbb{R}^{n}$ which is compact and has a piecewise smooth boundary. Let $\mathbf{F}: \Omega \rightarrow \mathbb{R}^{n}$ be a continuously differentiable vector field defined on a neighborhood of $\Omega$. If $\varphi: \Omega \rightarrow \mathbb{R}$ is a continuously differentiable vector field such that $\mathbf{F}=-\boldsymbol{\nabla} \varphi$, then the field $\mathbf{F}$ is said to be irrotational, that is $\nabla \wedge \mathbf{F}=0$.

Proof.
In index notation, we have

$$
(\boldsymbol{\nabla} \times \mathbf{F})_{i}=\epsilon_{i j k} \partial_{j} F_{k} \quad F_{k}=-\partial_{k} \varphi
$$

Substitution of the second relation into the first one gives

$$
\begin{equation*}
(\boldsymbol{\nabla} \times \mathbf{F})_{i}=-\epsilon_{i j k} \partial_{j} \partial_{k} \varphi \tag{I.2}
\end{equation*}
$$

Since the derivatives are commutative, it follows that $\partial_{j} \partial_{k}$ is a symmetric tensor in the indices $j, k$. On the other side, $\epsilon_{i j k}$ is completely antisymmetric. Then we will have, for all $i=1,2,3$ :

$$
\begin{equation*}
(\boldsymbol{\nabla} \times \mathbf{F})_{i}=-\epsilon_{i j k} \partial_{j} \partial_{k} \varphi=0 \tag{I.3}
\end{equation*}
$$

## Proposition 1.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ three vectors. Then the following identity holds:

$$
\begin{equation*}
\mathbf{a} \times(\mathbf{b} \times \mathbf{c})=(\mathbf{a} \cdot \mathbf{c}) \mathbf{b}-(\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \tag{I.4}
\end{equation*}
$$

Proof.
Let us use again the index notation:

$$
\begin{equation*}
(\mathbf{a} \times(\mathbf{b} \times \mathbf{c}))_{i}=\epsilon_{i j k} a_{j} \epsilon_{k l m} b_{l} c_{m}=\epsilon_{i j k} \epsilon_{k l m} a_{j} b_{l} c_{m} \tag{I.5}
\end{equation*}
$$

Observing that $\epsilon_{k l m}=\epsilon_{l m k}$ (even permutation) and using the identity $\epsilon_{i j k} \epsilon_{l m k}=\delta_{i l} \delta_{j k}-\delta_{i m} \delta_{l j}$, we can write

$$
\begin{align*}
& =\left(\delta_{i l} \delta_{j m}-\delta_{i m} \delta_{l j}\right) a_{j} b_{l} c_{m}=\delta_{i l} \delta_{j m} a_{j} b_{l} c_{m} \\
& -\delta_{i m} \delta_{l j} a_{j} b_{l} c_{m}= \\
& =b_{i} a_{j} c_{j}-c_{i} a_{j} b_{j}=(\mathbf{a} \cdot \mathbf{c}) b_{i}-(\mathbf{a} \cdot \mathbf{b}) c_{i} \tag{I.6}
\end{align*}
$$

From this identity, it follows that $\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{F})=$ $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F})-\nabla^{2} \mathbf{F}$, in fact

$$
\begin{align*}
\left(\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{F})_{i}\right. & =\epsilon_{i j k} \partial_{j} \epsilon_{k l m} \partial_{l} F_{m}= \\
& =\epsilon_{i j k} \epsilon_{l m k} \partial_{j} \partial_{l} F_{m}= \\
& =\left(\delta_{i l} \delta_{j m}-\delta_{i m} \delta_{j l}\right) \partial_{j} \partial_{l} F_{m}= \\
& =\partial_{j} \partial_{i} F_{j}-\partial_{j} \partial_{j} F_{i}= \\
& =\partial_{i} \partial_{j} F_{j}-\partial_{j} \partial_{j} F_{i}= \\
& =(\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}))_{i}-\nabla^{2} F_{i} \tag{I.7}
\end{align*}
$$

## II. RETARDED POTENTIALS

Maxwell's equations are a system of four partial differential equations:

$$
\begin{align*}
& \boldsymbol{\nabla} \cdot \mathbf{E}=\frac{\rho}{\epsilon_{0}} \\
& \boldsymbol{\nabla} \cdot \mathbf{B}=0 \\
& \boldsymbol{\nabla} \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} \\
& \boldsymbol{\nabla} \times \mathbf{B}=\mu_{0} \mathbf{J}+\frac{1}{c^{2}} \frac{\partial \mathbf{E}}{\partial t} \tag{II.1}
\end{align*}
$$

where

- $\mathbf{E}$ and $\mathbf{B}$ are respectively the electric field and magnetic field;
- $\rho$ and $\mathbf{J}$ are respectively the density of free charge and the density of free current;
- $\epsilon_{0}$ and $\mu_{0}$ are respectively the electric permittivity and magnetic permittivity in vacuum;
- $c=1 / \sqrt{\epsilon_{0} \mu_{0}}$ is the velocity of light in vacuum.

The first equation and the third equation link the fields to the respective sources (inhomogeneous equations), while the other two are homogeneous.
In order to solve the equations, we first try to decouple them: by taking the curl of the third equation and using the identity (1.1.9) we obtain

$$
\begin{align*}
\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \times \mathbf{E}) & =\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E})-\nabla^{2} \mathbf{E}= \\
& =\boldsymbol{\nabla} \times\left(\frac{\partial \mathbf{B}}{\partial t}\right)=\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) \tag{II.2}
\end{align*}
$$

Substitution the expressions for the divergence of $\mathbf{E}$ and for the curl of $\mathbf{B}$ by means of Maxwell's equations gives

$$
\begin{equation*}
\nabla^{2} \mathbf{E}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}=\frac{1}{\epsilon_{0} c^{2}} \frac{\partial \mathbf{J}}{\partial t}+\nabla \frac{\rho}{\epsilon_{0}} \tag{II.3}
\end{equation*}
$$

In this equation we recognize the typical form of the wave equation with a source term. An analogous algebra leads to obtain a similar equation for the magnetic field:

$$
\begin{equation*}
\nabla^{2} \mathbf{B}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{B}}{\partial t^{2}}=-\frac{1}{\epsilon_{0} c^{2}} \boldsymbol{\nabla} \times \mathbf{J} \tag{II.4}
\end{equation*}
$$

Although the equations for the fields are now decoupled, they are difficult to solve, because the source terms are complicated since they depend on the acceleration of them. Therefore, it is preferred to solve the equations in terms of the fields' potentials.
Since $\boldsymbol{\nabla} \cdot \mathbf{B}=0$, we can write the magnetic field as the curl of a vector field $\mathbf{A}$, called vector potential:

$$
\begin{equation*}
\mathbf{B}=\boldsymbol{\nabla} \times \mathbf{A} \tag{II.5}
\end{equation*}
$$

By substitution into the expression for the curl of the electric field, we find:

$$
\boldsymbol{\nabla} \times \mathbf{E}=-\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{A})=-\boldsymbol{\nabla} \times \frac{\partial \mathbf{A}}{\partial t}
$$

from which it follows

$$
\nabla \times\left(\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}\right)=0
$$

since that the vector in parenthesis has curl zero, it can be written as the gradient, with opposite sign, of a scalar field $\varphi$, called scalar potential. Ultimately, we have expressed the fields $\mathbf{E}$ and $\mathbf{B}$ as functions of the vector and scalar potentials:

$$
\begin{align*}
& \mathbf{B}=\boldsymbol{\nabla} \times \mathbf{A} \\
& \mathbf{E}=-\boldsymbol{\nabla} \varphi-\frac{\partial \mathbf{A}}{\partial t} \tag{II.6}
\end{align*}
$$

Putting this expressions in the inhomogeneous equations, we obtain:

$$
\begin{align*}
& \nabla^{2} \varphi+\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A}=-\frac{\rho}{\epsilon_{0}} \\
& \nabla^{2} \mathbf{A}-\nabla(\boldsymbol{\nabla} \times \mathbf{A})-\frac{1}{c^{2}} \frac{\partial}{\partial t} \nabla \varphi-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}}=-\frac{\mathbf{J}}{\epsilon_{0} c^{2}} \tag{II.7}
\end{align*}
$$

Even this equations seem to be quite complicated, but it is always possible, given the potentials, to effectuate a gauge transformation:

$$
\left\{\begin{array}{lll}
\mathbf{A} & \longrightarrow & \mathbf{A}^{\prime}=\mathbf{A}+\boldsymbol{\nabla} \Lambda  \tag{II.8}\\
\varphi & \longrightarrow & \varphi^{\prime}=\varphi-\frac{\partial \Lambda}{\partial t}
\end{array}\right.
$$

where $\Lambda \equiv \Lambda(\mathbf{x}, t)$ is an arbitrary scalar function of the coordinates and time. The fields are invariant under gauge transformations. As a matter of fact, we have

$$
\begin{equation*}
\mathbf{B}^{\prime}=\boldsymbol{\nabla} \times \mathbf{A}^{\prime}=\boldsymbol{\nabla} \times(\mathbf{A}+\boldsymbol{\nabla} \Lambda)=\boldsymbol{\nabla} \times \mathbf{A}=\mathbf{B} \tag{II.9}
\end{equation*}
$$

since the curl of a gradient is always zero. For the electric field we have instead:

$$
\begin{align*}
\mathbf{E}^{\prime} & =-\boldsymbol{\nabla} \varphi^{\prime}-\frac{\partial \mathbf{A}^{\prime}}{\partial t}=-\nabla \varphi+\nabla \frac{\partial \Lambda}{\partial t}-\frac{\partial \mathbf{A}}{\partial t}-\frac{\partial \boldsymbol{\nabla} \Lambda}{\partial t}= \\
& =-\boldsymbol{\nabla} \varphi-\frac{\partial \mathbf{A}}{\partial t}=\mathbf{E} \tag{II.10}
\end{align*}
$$

since, by virtue of the commutativity of derivatives, $\partial_{t} \boldsymbol{\nabla} \Lambda=\nabla \partial_{t} \Lambda$.
In our case, we will use the Lorenz's gauge, that is to choose $\Lambda$ in order to have

$$
\begin{equation*}
\boldsymbol{\nabla} \cdot \mathbf{A}=-\frac{1}{c^{2}} \frac{\partial \varphi}{\partial t} \tag{II.11}
\end{equation*}
$$

Using Lorenz's gauge, equations (1.1.18) will decouple:

$$
\begin{align*}
\nabla^{2} \varphi-\frac{1}{c^{2}} \frac{\partial^{2} \varphi}{\partial t^{2}} & =-\frac{\rho}{\epsilon_{0}}  \tag{II.12}\\
\nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}} & =-\frac{\mathbf{J}}{\epsilon_{0} c^{2}} \tag{II.13}
\end{align*}
$$

This time, the wave equations present a simpler source term and therefore the solution is easier. The form of the equations is

$$
\begin{equation*}
\nabla^{2} \psi(\mathbf{x}, t)-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \psi(\mathbf{x}, t)=-S(\mathbf{x}, t) \tag{II.14}
\end{equation*}
$$

In order to solve this equation, we use the superposition principle by decomposing the source $S$ as a sum of point sources placed in $\mathbf{x}_{0}$, that is

$$
\begin{equation*}
S(\mathbf{x}, t)=\int S\left(\mathbf{x}_{0}, t\right) \delta^{3}\left(\mathbf{x}-\mathbf{x}_{0}\right) \mathrm{d}^{3} x_{0} \tag{II.15}
\end{equation*}
$$

where $\delta^{3}\left(\mathbf{x}-\mathbf{x}_{0}\right)=\delta\left(x-x_{0}\right) \delta\left(y-y_{0}\right) \delta\left(z-z_{0}\right)$ is the three-dimensional generalization of Dirac's delta. Now we will search for a funtion $\psi_{x_{0}}$ (called Green functions or fundamental solution) which solves the equation
$\nabla^{2} \psi_{x_{0}}(\mathbf{x}, t)-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \psi_{x_{0}}(\mathbf{x}, t)=-S\left(\mathbf{x}_{0}, t\right) \delta^{3}\left(\mathbf{x}-\mathbf{x}_{0}\right)$
while the solution for the full equation will be given by

$$
\begin{equation*}
\psi(\mathbf{x}, t)=\int \psi_{x_{0}}\left(\mathbf{x}_{0}, t\right) \mathrm{d}^{3} x_{0} \tag{II.17}
\end{equation*}
$$

For $\neq \mathrm{x}_{\mathbf{0}}$ there is no source and the equation is homogeneous. Searching for solutions with spherical simmetry, we obtain, setting $r=\left|\mathbf{x}-\mathbf{x}_{0}\right|$ :

$$
\begin{equation*}
\psi_{x_{0}}=\frac{f(t-r / c)}{r}+\frac{g(t+r / c)}{r} \tag{II.18}
\end{equation*}
$$

The causality principle, thought, provides the constrain that the waves are outgoing from the sources (moreover, it has little physical sense to talk about waves that come from infinity to the source), then our solution and its gradient will respectively be:

$$
\begin{align*}
\psi_{x_{0}} & =\frac{f(t-r / c)}{r} \\
\nabla \psi_{x_{0}} & =-\frac{f(t-r / c)}{r^{2}}-\frac{1}{c} \frac{f^{\prime}(t-r / c)}{r} \tag{II.19}
\end{align*}
$$

We integrate now equation (1.3.16) over a sphere of radius $R$ and centre in $\mathbf{x}_{0}$ and substitute the expression just found for the gradient of $\psi_{x_{0}}$ :

$$
\begin{align*}
& \int_{B_{R}\left(\mathbf{x}_{0}\right)} \boldsymbol{\nabla} \cdot\left[-\frac{f}{r^{2}}-\frac{1}{c} \frac{f^{\prime}}{r}\right] \mathrm{d}^{3} x-\frac{1}{c^{2}} \int_{B_{R}\left(\mathbf{x}_{0}\right)} \frac{f^{\prime \prime}}{r} \mathrm{~d}^{3} x= \\
& =-\int_{B_{R}\left(\mathbf{x}_{0}\right)} S\left(\mathbf{x}_{0}, t\right) \delta^{3}\left(\mathbf{x}-\mathbf{x}_{0}\right) \mathrm{d}^{3} x \tag{II.20}
\end{align*}
$$

For $R \rightarrow 0$, we note that the second term of the left side tends to zero when integrated over the sphere. Applying Gauss theorem to the first integral we obtain

$$
\begin{equation*}
\int_{B_{R}\left(\mathbf{x}_{0}\right)} \nabla \cdot\left[-\frac{f}{r^{2}}-\frac{1}{c} \frac{f^{\prime}}{r}\right] \mathrm{d}^{3} x=4 \pi R^{2}\left[-\frac{f}{R^{2}}-\frac{1}{c} \frac{f^{\prime}}{R}\right] \tag{II.21}
\end{equation*}
$$

Therefore in the limit $R \rightarrow 0$ we remain with only the first term. Then we find the following equality:

$$
\begin{equation*}
-4 \pi f=-S\left(\mathbf{x}_{0}, t\right) \quad \Longrightarrow \quad f(t)=\frac{S\left(\mathbf{x}_{0}, t\right)}{4 \pi} \tag{II.22}
\end{equation*}
$$

The boundary condition for $r \rightarrow 0$ is

$$
\begin{equation*}
\psi_{x_{0}}=\frac{f(t)}{r}=\frac{S\left(\mathbf{x}_{0}, t\right)}{4 \pi r} \tag{II.23}
\end{equation*}
$$

from which we have

$$
\begin{equation*}
\psi_{x_{0}}(r, t)=\frac{S\left(\mathbf{x}_{0}, t-r / c\right)}{4 \pi r} \tag{II.24}
\end{equation*}
$$

We note that it appears a retarded time $t_{\text {rit }}=t-r / c$ in the argument of the source. Having solved the equation for a point source, we need just to integrate over $\mathbf{x}_{0}$ in order to find the full solution:

$$
\begin{equation*}
\psi(\mathbf{x}, t)=\int \frac{S\left(\mathbf{x}_{0}, t-\frac{\left|\mathbf{x}-\mathbf{x}_{0}\right|}{c}\right)}{4 \pi r} \mathrm{~d}^{3} x_{0} \tag{II.25}
\end{equation*}
$$

The solutions to Maxwell's equations (the retarded potentials) are therefore given by

$$
\begin{aligned}
& \mathbf{A}(\mathbf{x}, t)=\frac{1}{4 \pi \epsilon_{0} c^{2}} \int \frac{\mathbf{J}\left(\mathbf{x}_{0}, t-\frac{\left|\mathbf{x}-\mathbf{x}_{0}\right|}{c}\right)}{\left|\mathbf{x}-\mathbf{x}_{0}\right|} \mathrm{d}^{3} x_{0} \\
& \varphi(\mathbf{x}, t)=\frac{1}{4 \pi \epsilon_{0}} \int \frac{\rho\left(\mathbf{x}_{0}, t-\frac{\left|\mathbf{x}-\mathbf{x}_{0}\right|}{c}\right)}{\left|\mathbf{x}-\mathbf{x}_{0}\right|} \mathrm{d}^{3} x_{0}
\end{aligned}
$$

The fields are obviously related to the potentials by the equations

$$
\begin{align*}
& \mathbf{E}(\mathbf{x}, t)=-\boldsymbol{\nabla} \varphi(\mathbf{x}, t)-\frac{\partial}{\partial t} \mathbf{A}(\mathbf{x}, t) \\
& \mathbf{B}(\mathbf{x}, t)=\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}, t) \tag{II.27}
\end{align*}
$$

## III. MULTIPOLES EXPANSION

After having found the expressions for the retarded potentials, a reflection about the retarded
time $t-r / c$ is due. Let us consider a given distribution of charge and current, namely $\rho, \mathbf{J}$; let $\mathbf{x}^{\prime} \equiv\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$ the coordinates of a point inside the source. We now want to evaluate the potentials at $\mathbf{r} \equiv(x, y, z)$ (let $d$ be the characteristic dimension of the source). In the electrostatic treatment, we considered points placed at a distance $r$ such that $r \gg d$ and expanded the electrostatic potential in series of $1 /\left|\mathbf{r}-\mathbf{x}^{\prime}\right|$. In this case, the term $\left|\mathbf{r}-\mathbf{x}^{\prime}\right|$ appears also in the argument of the source, and this complicates the evaluation. Let us consider for example the vector potential:

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t)=\frac{1}{4 \pi \epsilon_{0} c^{2}} \int \frac{\mathbf{J}\left(\mathbf{x}^{\prime}, t-\left|\mathbf{r}-\mathbf{x}^{\prime}\right| / c\right)}{\left|\mathbf{r}-\mathbf{x}^{\prime}\right|} \mathrm{d}^{3} x^{\prime} \tag{III.1}
\end{equation*}
$$

In the hypothesis $d \ll r$, we can approximate $1 /\left|\mathbf{r}-\mathbf{x}^{\prime}\right| \simeq 1 / r$, obtaining

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t) \simeq \frac{1}{4 \pi \epsilon_{0} c^{2} r} \int \mathbf{J}\left(\mathbf{x}^{\prime}, t-\left|\mathbf{r}-\mathbf{x}^{\prime}\right| / c\right) \mathrm{d}^{3} x^{\prime} \tag{III.2}
\end{equation*}
$$

Now we need to find some conditions for which it is possible to expand the retarded time; the condition that is found is (we omit the calculations) $d / \lambda \ll 1$, where $\lambda$ is the wavelenght of the produced wave. Therefore, in the hypothesis $d \ll r, d \ll \lambda$ it is possible to expand $\mathbf{J}$ :

$$
\begin{align*}
& \mathbf{J}\left(\mathbf{x}^{\prime}, t-\left|\mathbf{r}-\mathbf{x}^{\prime}\right| / c\right) \simeq \\
& \simeq \mathbf{J}\left(\mathbf{x}^{\prime}, t-r / c\right)+\frac{\mathbf{x}^{\prime} \cdot \mathbf{r}}{c r} \frac{\partial \mathbf{J}}{\partial t}\left(\mathbf{x}^{\prime}, t-r / c\right)+\cdots \tag{III.3}
\end{align*}
$$

obtaining then the following expression for $\mathbf{A}$ :

$$
\begin{align*}
& \mathbf{A}(\mathbf{r}, t) \simeq \mathbf{A}_{1}+\mathbf{A}_{2}= \\
& =\frac{1}{4 \pi \epsilon_{0} c^{2}} \frac{1}{r} \int \mathbf{J}\left(\mathbf{x}^{\prime}, t-r / c\right) \mathrm{d}^{3} x+ \\
& +\frac{1}{4 \pi \epsilon_{0} c^{2}} \frac{1}{r} \int \frac{\mathbf{x}^{\prime} \cdot \mathbf{r}}{c r} \frac{\partial \mathbf{J}}{\partial t}\left(\mathbf{x}^{\prime}, t-r / c\right) \mathrm{d}^{3} x^{\prime} \tag{III.4}
\end{align*}
$$

Let us briefly examine the term $\mathbf{A}_{1}$, which represents the vector potential in electric dipole approximation. From the identity

$$
\begin{equation*}
\mathbf{J}=\boldsymbol{\nabla}^{\prime} \cdot\left(\mathbf{x}^{\prime} \wedge \mathbf{J}\right)-\mathbf{x}^{\prime} \boldsymbol{\nabla}^{\prime} \cdot \mathbf{J} \tag{III.5}
\end{equation*}
$$

where $\boldsymbol{\nabla}^{\prime}=\left(\partial_{x^{\prime}}, \partial_{y^{\prime}}, \partial_{z^{\prime}}\right)$, we have

$$
\begin{equation*}
\int \mathbf{J} \mathrm{d}^{3} x^{\prime}=\int \boldsymbol{\nabla}^{\prime} \cdot\left(\mathbf{x}^{\prime} \wedge \mathbf{J}\right) \mathrm{d}^{3} x^{\prime}-\int \mathbf{x}^{\prime} \boldsymbol{\nabla}^{\prime} \cdot \mathbf{J} \mathrm{d}^{3} x^{\prime} \tag{III.6}
\end{equation*}
$$

By applying Gauss theorem, the first term of the right side becomes a flux term that cancels out by
choosing a surface that contains the entire distribution of current; the second term can be written, using the continuity equation, as:

$$
\begin{align*}
\int \mathbf{J} \mathrm{d}^{3} x^{\prime} & =-\int \mathbf{x}^{\prime} \nabla^{\prime} \cdot \mathbf{J} \mathrm{d}^{3} x^{\prime}=\int \mathbf{x}^{\prime} \frac{\partial \rho}{\partial t} \mathrm{~d}^{3} x^{\prime}= \\
& =\frac{\mathrm{d}}{\mathrm{~d} t} \int \mathbf{x}^{\prime} \rho \mathrm{d}^{3} x^{\prime}=\dot{\mathbf{p}} \tag{III.7}
\end{align*}
$$

where $\mathbf{p}$ is the electric dipole moment of the distribution of the sources. Therefore we obtain

$$
\begin{equation*}
\mathbf{A}_{1}(\mathbf{r}, t)=\frac{1}{4 \pi \epsilon_{0} c^{2}} \frac{\dot{\mathbf{p}}(t-r / c)}{r} \tag{III.8}
\end{equation*}
$$

The relative scalar potential can be calculated from Lorenz's gauge:

$$
\begin{equation*}
\frac{1}{c^{2}} \frac{\partial \varphi_{1}}{\partial t}=\nabla \cdot \mathbf{A}_{1} \Longrightarrow \varphi_{1}(\mathbf{r}, t)=\frac{1}{4 \pi \epsilon_{0}} \frac{\mathbf{r} \cdot\left[\mathbf{p}+\frac{r}{c} \dot{\mathbf{p}}\right]_{t-\frac{r}{c}}}{r^{3}} \tag{III.9}
\end{equation*}
$$

## A. Expressions for the fields

From the expressions that relate the fields to the potential we immediately have

$$
\begin{aligned}
\mathbf{B}_{1} & =\boldsymbol{\nabla} \times \mathbf{A}_{1}=\frac{1}{4 \pi \epsilon_{0} c^{2}} \frac{\left[\dot{\mathbf{p}}+\frac{r}{c} \ddot{\mathbf{p}}\right]_{t-\frac{r}{c}} \times \mathbf{r}}{r^{3}} \\
\mathbf{E}_{1} & =-\boldsymbol{\nabla} \varphi_{1}-\frac{\partial \mathbf{A}_{1}}{\partial t}= \\
& =-\frac{1}{4 \pi \epsilon_{0} r^{3}} \underbrace{\left[\mathbf{p}^{*}-3 \frac{\mathbf{p}^{*} \cdot \mathbf{r}}{r^{2}}\right.}_{\text {close field }}-\underbrace{\left.\frac{1}{c^{2}}\left(\ddot{\mathbf{p}}_{t-r / c} \times \mathbf{r}\right) \times \mathbf{r}\right]}_{\text {radiation field }}
\end{aligned}
$$

(III.10)
where $\mathbf{p}^{*}=\mathbf{p}(t-r / c)+\frac{r}{c} \dot{\mathbf{p}}(t-r / c)$. All we have discussed until this point holds in the limits $d / r \ll 1, d / \lambda \ll 1$. The parameter $r / \lambda$ is left to discuss; according to the values that this parameter takes, we can distinguish two zones: $r / \lambda \ll 1$, close field zone. The fields are proportional to $1 / r^{2}$ and depend on the velocity of the sources; $r / \lambda \gg 1$, radiation field zone. The fields are proportional to $1 / r$ and depend on the acceleration of the sources. In conclusion, the retarded time $t-r / c$ that appears in the solution of D'Alembert equation slightly complicated the evaluation of the multipoles expansion compared to the electrostatic case, where the equation to solve was Poisson's equation only for the scalar potential.


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