Mathematical Methods for Physics

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Contents

1 Introduction: algebraic structures

Most of this course will be devoted to topics in algebra. Loosely speaking, algebra studies relationships among operations. In particular, certain notable structures have emerged over time, from the study of concrete cases. Unfortunately, often these structures are introduced by giving arid lists of axioms. I will try to introduce them here in a slightly informal way, highlighting the relationships among them whenever possible. We start from a set, and we gradually add "operations" among the elements of the set.

Our first definition consists in adding a "composition" to a set:

Definition 1.1. *A* ***group*** *is a set G endowed with a* composition $\circ : G \times G \to G$ *(namely, $g_1 \circ g_2 \in G$) such that*

- *The composition is* associative*: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) (\equiv g_1 \circ g_2 \circ g_3)$.*
- *There exists an* identity element *e, such that $e \circ g = g \circ e = g$, $\forall g$.*
- *The* inverse *of any element exists: $\forall g$, $\exists g^{-1}$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.*

We will sometimes omit $\circ$, and simply write the elements one after another: $g_1 g_2 \equiv g_1 \circ g_2$.

The concept of group is probably the most important one in the entire course.

Example 1.1. *You have already seen many examples of groups.*

1. *The real numbers $\mathbb{R}$ are a group, with composition $\circ = +$. The identity element is 0. The inverse of a number x is $-x$. The same is true for the set of relative numbers $\mathbb{Z}$. On the contrary, the set of natural numbers $\mathbb{N}$ is not a group: there is no inverse (the inverse of n under $\circ = +$ would be $-n$).*

2. *The set of positive real numbers $\mathbb{R}_+$ is a group with composition $\circ = \cdot$. The set of natural numbers $\mathbb{N}$ is not: there is no inverse (the inverse of n under $\circ = \cdot$ would be $1/n$).*

3. *Let us consider the group $\mathbb{Z}$, and let us introduce an equivalence relation $\sim$, defined by $n + p \sim n$, for a certain fixed integer p. We will denote by 0 the equivalence class $\{0, p, 2p \ldots\}$, by 1 the equivalence class $\{1, 1+p, 1+2p \ldots\}$, and so on. The resulting quotient is denoted $\mathbb{Z}/p\mathbb{Z}$, or more simply $\mathbb{Z}_p$; it is called "cyclic group with p elements", because there are p equivalence classes: $0,\ 1, \ldots, p-1$.*

4. *The set of square matrices with real entries,* $\mathrm{Mat}(N,\mathbb{R})$*, or with complex entries,* $\mathrm{Mat}(N,\mathbb{C})$*, both have a natural composition: the usual matrix multiplication. As you learned in school, this composition is associative; the identity element is nothing but the identity matrix* 1_N*. However, the inverse need not exist. Consider now* $\mathrm{Gl}(N,\mathbb{R})$ *or* $\mathrm{Gl}(N,\mathbb{C})$*, the set of* invertible *matrices with real or complex entries. Within these sets, the inverse exists by definition; hence, these sets are groups.*

The **order** of a group G, denoted by $\#(G)$, is the number of its elements. A *finite* group has a finite number of elements: an example is $\mathbb{Z}_p$, which we introduced just above. We will first study finite groups, which have applications in molecular physics and crystallography. We will then consider infinite groups, which are of paramount importance in high-energy physics.

Even among infinite groups, there seems to be some fundamental difference between "discrete" groups, such as $\mathbb{Z}$, and "continuous" group, such as $\mathbb{R}$. This distinction can be formalized only after having introduced some additional structure, namely a topology. We will not comment on this for the time being.

Another important distinction is the following:

Definition 1.2. *A group* G *is* abelian *if* $g_1 \circ g_2 = g_2 \circ g_1$, $\forall g_1, g_2 \in G$*. It is* nonabelian *otherwise.*

Example 1.2. 1. $\mathbb{Z}$, $\mathbb{R}$ *and* $\mathbb{Z}_p$ *are abelian groups with respect to* $\circ = +$.

2. $\mathrm{Gl}(n,\mathbb{C})$ *and* $\mathrm{Gl}(n,\mathbb{R})$ *are nonabelian groups.*

We will sometimes use the symbol "+" for the composition in an abelian group, to stress that it is commutative.

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Since this is our first lecture, I will pause to introduce other algebraic structures, as I promised. We will come back to groups in the next section.

The first generalization consists in adding a second composition.

Definition 1.3. *A* ***ring*** R *is an abelian group, whose composition we will call* +[1]*, endowed with a second composition* $\cdot : R \times R \to R$*, such that:*

- *The second opearation* $\cdot$ *is associative:* $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$.

[1] The inverse of r will be called $-r$; the identity element with respect to $+$ will be called 0.

- *Distributive properties:* $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$, *e* $(r_1 + r_2) \cdot r_3 = r_1 \cdot r_3 + r_2 \cdot r_3$.

Notice that the second operation $\cdot$ does not have all the properties we demanded for the composition of a group. It is associative, but neither the identity element nor the inverse need exist. In fact, demanding that 0 have an inverse with respect to the second composition $\cdot$ does not seem a particularly good idea. But:

Definition 1.4. *A **field** $\mathbb{F}$ is a ring such that $\mathbb{F} - \{0\}$ (where 0 is the identity element with respect to the first composition +; see footnote 1) is an abelian group also with respect to the second composition $\cdot$. (Hence: $\exists$ 1 such that $1 \cdot f = f \cdot 1 = f$, $\forall f$; and, $\forall f \neq 0$, $\exists f^{-1}$ such that $f \cdot f^{-1} = f^{-1} \cdot f = 1$).*

Example 1.3. *You also know already lots of examples of rings and of fields.*

1. *We have seen that $\mathbb{Z}$ is an abelian group with respect to $\circ = +$. It is also a ring, if we take the second composition $\cdot$ to be the ordinary multiplication of integers. Indeed, multiplication is associative, and the distributive properties are those you learned in elementary school. It is not a field, because the inverse of an integer with respect to $\cdot$ is not an integer.*

2. *$\mathbb{R}$ is also a ring, if we take + and $\cdot$ as first and second composition. It is also a field.*

3. *$\mathbb{C}$ is a field too.*

4. *Let us now consider $\mathbb{Z}_p$. **Theorem**: if p is* prime*, $\mathbb{Z}_p$ is a field! (Exercise: try to find the inverse with respect to $\cdot$ of all non-zero elements in $\mathbb{Z}_5$.)*

5. *The space of polynomials $p(x)$ of one variable with real coefficients is a ring; again, you have learned the two compsitions + and $\cdot$ in elementary school.*

6. *Let us define* $\mathrm{Mat}(N, \mathbb{F})$ *as the set of $N \times N$ matrices whose entries are in $\mathbb{F}$. If $\mathbb{F}$ is a field, we can also define* $\mathrm{Gl}(N, \mathbb{F})$*, the set of* invertible *matrices whose entries are in $\mathbb{F}$. (If $\mathbb{F}$ is not a field, this doesn't even work for $N = 1$.) Once again, this is a group.*

Let us now go back to abelian groups. Rather than adding a new operation, as we have done so far, suppose we can "multiply" by quantities which are "external" to the group.

Definition 1.5. *A **module** M with respect to a ring R (also: an R-module) is an abelian group, endowed with an "action" $\cdot : R \times M \to M$ (that is: $r \cdot m \in M$, $\forall r \in R$, $\forall m \in M$)*[2]*, such that:*

1. *Associativity:* $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$, $\forall r_1, r_2 \in R$, $\forall m \in M$.

2. *Distribution laws:* $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$, $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$, $\forall r, r_1, r_2 \in R$, $\forall m, m_1, m_2 \in M$.

This definition is a generalization of the concept of ring, as you can see by going back to definition 1.3. In particular, R can be considered a module over itself!

Exercise: A $\mathbb{Z}$-module is... an abelian group.

We already know some modules. If we take a ring that is also a field:

Definition 1.6. *A **vector space** V with respect to a field $\mathbb{F}$ is a module over $\mathbb{F}$ such that the identity $1 \in \mathbb{F}$ acts as the identity on V: in other words, $1 \cdot v = v$, $\forall v \in V$.*

We can finally put together the structures of ring and of module:

Definition 1.7. *An **associative algebra** A over a ring R is a ring which is also a module with respect to R, such that the action of R on A is bilinear: namely, $r \cdot (a_1 \cdot a_2) = (r \cdot a_1) \cdot a_2 = a_1 \cdot (r \cdot a_2)$.*

So this is more than a ring because we are now allowed to take R-multiples.

Exercise: a $\mathbb{Z}$-algebra is... a ring.

Example 1.4. *The space of matrices $\mathrm{Mat}(N, F)$ is an algebra over F (**exercise**: check this.)*

2 Introduction to groups

We now give a first look to groups. We are going to learn when two groups are "equal" (isomorphic), how to realize the elements of a group as "transformations" (representations), how to make a group from two smaller groups (direct and semidirect product), and how to find groups within other groups (subgroups). We will also see more examples of groups.

[2] We will call $+$ both the group composition in M and the one in R. We will call $\cdot$ both the action $R \times M \to M$ and the second composition in R. This is standard, and should not create confusion. As an exercise, every time you see a $+$ or a $\cdot$, ask yourself which operation we are talking about.

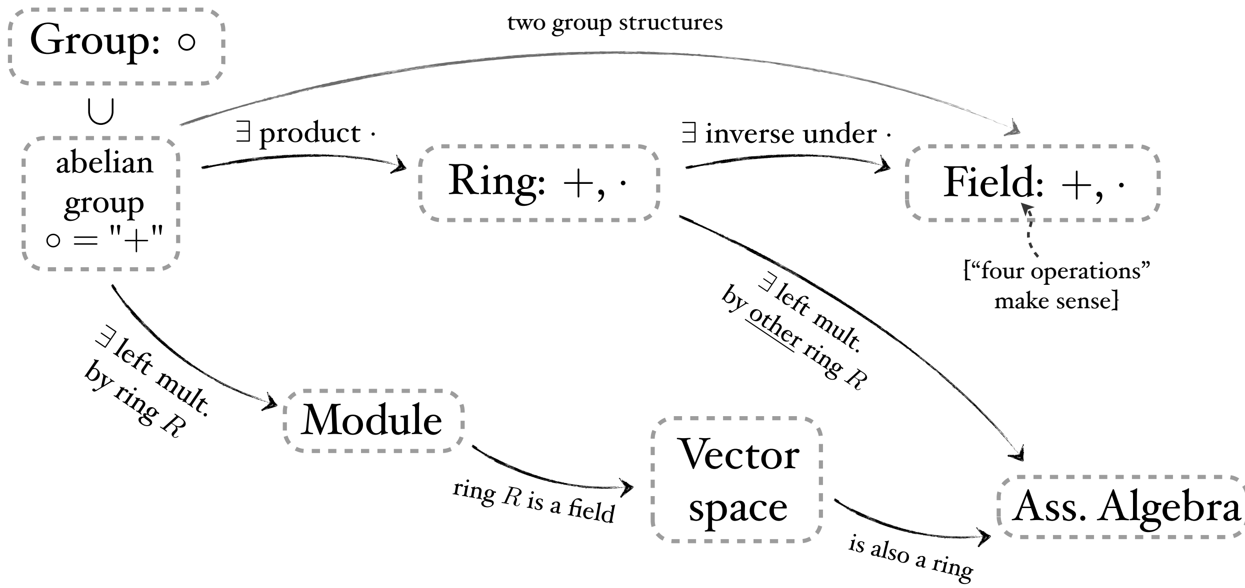


Figure 1: A summary of the algebraic structures in section 1.

2.1 Presentations

We now introduce a peculiar class of examples:

Definition 2.1. *Given a set S, let $\langle S\rangle = \{s_1^{a_1} s_2^{a_2} \dots s_k^{a_k},\ k \in \mathbb{Z}, a_i \in \mathbb{Z}\ , s_i \in S\}/\sim$, be the set of "words" of arbitrary length that one can form by concatenating elements of S, with an equivalence relation $s^i s^j \sim s^{i+j}$. This is called a* ***free group****. The composition is just given by concatenating two words in a longer word. The identity is the empty word, where all the $a_i = 0$. [Exercise: find the inverse element of a given word.]*

Example 2.1.

- *If we have only one generator r, we can only form words of the type r^i, $i \in \mathbb{Z}$. Because of the equivalence relation $\sim$, the product of two such words gives $r^i r^j = r^{i+j}$.*

  Intuitively, this group looks "just like" the group of the integers $\mathbb{Z}$ we saw above (by having r^i correspond to i). The right word is "isomorphism", and we will introduce it in section 2.2.

- *If we have two generators, say r and s, there are infinitely many words we can form: r, s, $rs^2r^{-1}s^{17}$, … We have not seen this group before; it is simply called "the free group with two generators".*

This class of examples might not seem very useful. However, it gives rise to the following useful way of defining a group:

Definition 2.2. *Let $R \subset \langle S \rangle$ be a certain subset of words in $\langle S \rangle$. A* ***presentation*** *$\langle S|R \rangle$ is then the group defined by the equivalence classes in $\langle S \rangle$ of the equivalence relation $r \equiv e$,$\forall r \in R$; e is the identity (the null word). S is called the set of* ***generators****; R is called the set or* ***relations****.*

Example 2.2. *Let us consider again a single generator, $S = \{r\}$, but this time let us introduce a relation: $R = \{r^k\}$. This means that whenever we see r^k, we can identify it with the identity e. Again, intuitively this looks like it should be "the same as" the group $\mathbb{Z}_k$ we saw earlier; the right word, which we will introduce in section 2.2, is that they are "isomorphic".*

2.2 Homomorphisms, isomorphisms, representations

Definition 2.3. *A homomorphism ϕ from a group G to another group $\tilde{G}$ is a map $\phi : G \to \tilde{G}$ such that*

$$\phi(g_1 \circ g_2) = \phi(g_1) \circ \phi(g_2) \ . \tag{2.2.1}$$

(Notice that $\circ$ on the left hand side is the composition in G, and $\circ$ on the right hand side is the composition in $\tilde{G}$.)

This means, more or less, that ϕ "respects" the structures of group on G and $\tilde{G}$.

The problem with the concept of homomorphism is that it can "forget" some of the elements. For example, let us introduce the group G_{Id} whose only element is the identity, and let us define the map ϕ_{Id} that sends another G in G_{Id}: $\phi_{\text{Id}}(g) = e$, $\forall g \in G$. This is a homomorphism, but it "forgets" the whole structure of G. At the other extreme, we have the homomorphisms that "don't forget anything":

Definition 2.4. *An* ***isomorphism*** *$\phi : G \to \tilde{G}$ is an invertible homomorphism. (In other words, there exists a second homomorphism $\phi^{-1} : \tilde{G} \to G$ such that $\phi^{-1}(\phi(g)) = g$, $\forall g \in G$.) If $G = \tilde{G}$, ϕ is called an* ***automorphism****.*

We saw already examples of isomorphisms in examples 2.1 and 2.2. We will see many more later in this course.

Let us now see some additional examples of groups.

Example 2.3. *Let us define a group by giving its presentation (see definition 2.2). Consider the set S of two generators — call them r and σ; and a set of relations $R = \{r^k, \sigma r \sigma r, \sigma^2\}$. In other words, we are imposing*

$$r^k = e \ , \qquad \sigma r \sigma = r^{-1} \ , \qquad \sigma^2 = e \ , \tag{2.2.2}$$

where e is the identity. This is called the dihedral group.

This name suggests a geometrical origin for this group. Indeed we can think of D_k as the group of symmetries of a regular polygon with k sides. If for example we picture the polygon as having at least one horizontal side, r can be thought of as a rotation by an angle $2\pi/k$, while σ is a reflection around the vertical axis; see figure 2. [Exercise: find all the elements, and show that there are $2k$ of them.]

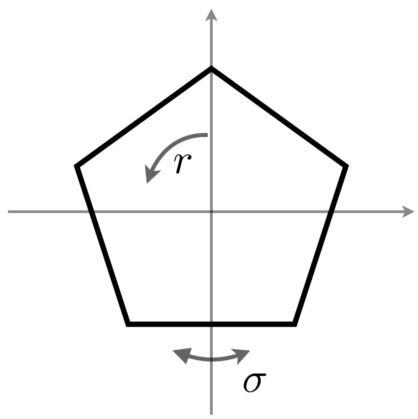


Figure 2: The dihedral group D_k can be seen as the group of symmetries of a k-gon. Here $k = 5$: r can be thought of as a rotation by $2\pi/5$, and σ as a reflection.

Example 2.4. *Recall that an $N \times N$ matrix O is called* orthogonal *if*[3]

$$OO^t = O^tO = 1_N \ . \tag{2.2.3}$$

This definition makes sense for matrices with entries in any field $\mathbb{F}$. The set of $N \times N$ orthogonal matrices $\mathrm{O}(N, \mathbb{F})$ is a group. $\mathrm{O}(N, \mathbb{R})$ is often called more simply $\mathrm{O}(N)$.

Also this group can be thought in a geometrical way: $\mathrm{O}(N)$ are the rotations *in N dimensions.*

*Notice that (2.2.3) implies that $\det(O)^2 = 1$, which means $\det(O) = \pm 1$. The set of orthogonal matrices with $\det(O) = 1$ is a group (**exercise**: check this); it is denoted by $SO(N)$ (*special orthogonal *group).*

[3] It is enough to impose $OO^t = 1_N$; from this it follows that $O^tO = 1_N$. A similar remark applies to (2.2.4) below.

Example 2.5. *Recall also that a matrix U is called* unitary *if*

$$UU^\dagger = U^\dagger U = 1_N \ . \tag{2.2.4}$$

The set of complex unitary $N \times N$ matrices $\mathrm{U}(N)$ *is also a group.*

The two previous examples show that the abstract definition of group has a concrete geometrical origin as a set of "transformations" of geometrical objects. The concept of group encodes the "relations" among these transformations. These relations can then be "represented" as geometrical tranformations in very different ways, according to the context.

More formally:

Definition 2.5. *A **representation** ρ of a group G is a homomorphism $\rho : G \to \mathrm{Gl}(N, \mathbb{C})$. N is called* dimension *of the representation.*

We can also think of ρ as a map $\rho : G \times V \to V$, where $V = \mathbb{C}^N$.

Just like we formalized when two groups are "the same", by defining isomorphisms, we also want to be able to recognize when two representations are the same. The definition is obvious:

Definition 2.6. *Two representations ρ_1 and ρ_2 are **equivalent** if one can be brought into another by a change of basis: namely, there exists an invertible T such that*

$$\rho_1(g) = T^{-1}\rho_2(g)T \ , \qquad \forall g \in G \ . \tag{2.2.5}$$

Given a presentation $\langle S|R\rangle$ of a group, we can give a representation by choosing $\rho(s)$ for all $s \in S$, such that $\rho(r) = 1$ for any relation $r \in R$. Let us see some example:

Example 2.6.

- *A representation for the group $\mathbb{Z}_k$ can be given by a matrix $\rho(r)$ such that $\rho(r)^k = \rho(r^k) = \rho(1) = 1$. We can easily an example of one-dimensional complex representation this way:*

$$\rho(r) = e^{2\pi i/k} \equiv \omega_k \ . \tag{2.2.6}$$

- *When we introduce the dihedral group D_k in example 2.3, we described already a representation, of dimension 2. Let us now write explicitly the matrices that represent the generators, in the case D_3:*

$$\rho(r) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \ , \qquad \rho(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{2.2.7}$$

It is easy to check that $\rho(r^3) = \rho(r)^3 = 1$ and that $\rho(\sigma^2) = \rho(\sigma)^2 = 1$, as it should be. The check works because $\rho(r)$ is simply a rotation by $2\pi/3$, and $\rho(\sigma)$ is a reflection about the vertical axis.

Example 2.7. *We implicitly saw another example of representation when introducing the orthogonal group in example 2.4. This is because we introduced the group already in terms of $N \times N$ matrices O. So a possible representation of an element $O \in \mathrm{O}(N)$ consists in just taking O itself! this acts on $\mathbb{R}^N$, and so we have a representation of dimension N, called the* ***fundamental representation****. Similar considerations apply to the unitary group $\mathrm{U}(N)$.*

Notice that our definition of representation does not exclude that an element might be "represented" trivially as the identity element. An extreme case is

$$\rho_1 : G \to \mathrm{Gl}(1, \mathbb{C}) \ , \qquad \rho_1(g) \equiv 1 \ \ \forall g \in G \ , \tag{2.2.8}$$

namely the homomorphism which maps any g into 1, thought of as a 1×1 matrix. (This is very similar to the trivial isomorphism ϕ_{Id} we saw after definition 2.3.) Clearly this "trivial representation" is not very interesting. At the other extreme, we find "faithful representations", namely the ones which are injective.

How many representations exist for a given group? this question will stay with us until the end of this course. As we will see, in many cases one can give an exhaustive classification.

As a side remark, the idea of representation is reminiscent of the one of module. Indeed, ρ gives us a way to multiply an element of G by an element of the vector space $V = \mathbb{C}^N$; in other words a map $G \times V \to V$, which associates to (g, v) the element $\rho(g)v \in V$. This is similar to the map $\cdot : R \times M \to M$ in the definition of a module. Some mathematicians actually call a representation of G a "G-module".

We now come to another example of group, whose representations might be less obvious to find.

Example 2.8. *The* permutation group *of k elements, S_k, is the group of bijective maps of the set $\{1, 2, \ldots, k\}$ to itself.*

This group has $k!$ elements. An example of an element in S_6 is

$$\sigma_{361254} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 2 & 5 & 4 \end{pmatrix} . \tag{2.2.9}$$

In this notation, the upper row gets mapped to the lower one: 1 *is mapped to* 3, 2 *to* 6*, and so on. In other words,*

$$\sigma_{\{n_i\}}(i) = n_i \ . \tag{2.2.10}$$

A notation more efficient than (2.2.9) can be obtained by decomposing our map in "cycles". Notice for example that 1 is mapped to 3, and 3 to 1. This is a "cycle" of length 2, or a 2-cycle. Let us now see what happens to 2: it gets mapped to 6, which gets mapped to 4, which in turn gets mapped to 2. This is a 3-cycle. Finally, 5 is left alone: it is a trivial cycle (or a 1-cycle). We can rewrite the (2.2.9) by listing all its cycles:

$$\sigma_{361254} = (13)(264) \ . \tag{2.2.11}$$

(We omitted the 1-cycle (5)*.)*

Let us also compute some compositions in this notation. Let us now consider S_5*, and consider* $(12) \circ (12345) \circ (12)$*, using the notation in (2.2.11). 1 gets sent to 2 by the first group element* (12)*, which is sent to 3 by the second group element* (12345)*, which is left undisturbed by the third group element* (12)*. So in total 1 is sent to 3. 3 is left undisturbed by the first element, which is sent to 4 by the second element, which is left alone by the third. So in total 3 is sent to 4. Continuing in this way, we obtain (**exercise**: check this)*

$$(12) \circ (12345) \circ (12) = (21345) \ . \tag{2.2.12}$$

How shall we represent S_k? we could, for example, act by permuting the elements of a basis. For example, the element (2.2.9) $\in S_6$ would be represented by the permutation matrix

$$\rho_{\text{perm}}(\sigma_{361254}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \ ; \tag{2.2.13}$$

look at its action on the basis vectors e_i. More generally, let us define:

$$\rho_{\text{perm}}(\sigma_{\{n_i\}})(e_i) = e_{n_i} \ . \tag{2.2.14}$$

(It is just a fancy way of describing a matrix such as (2.2.13)).

This representation seems quite 'big'; it is natural to wonder whether one can find others, in vector spaces of smaller dimension.

Definition 2.7. *The* direct sum $\rho = \rho_1 \oplus \rho_2$ *of two representations* ρ_1, ρ_2 *is defined by*

$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix} \tag{2.2.15}$$

$\forall g \in G$. *A representation* $\rho : G \times V \to V$ *is called* reducible *if it can be written as a direct sum of two representations (as in (2.2.15)) by a change of basis — or in other words, if it is equivalent to a direct sum (using definition 2.6). Equivalently:* ρ *is reducible if* V *splits in two subspaces* (V_1, $V_2 \subset V$ *such that* $V = V_1 \oplus V_2$) *which are invariant under* ρ*: namely,* $\rho(g)V_1 = V_1$, $\rho(g)V_2 = V_2$ $\forall g \in G$.

A representation is also called decomposable *if, again by a change of basis, it can be brought in a block-triangular form. In other words:* ρ *is decomposable if there exists a subspace* $V_1 \subset V$ *(one which is nontrivial: namely,* $V_1 \neq \emptyset, V$*) invariant under all the* $\rho(g)$*. (Notice that a reducible representation is also decomposable.)*

Finally, a representation is called ***irreducible*** *if it is not decomposable: in other words, if there is* ***no*** *nontrivial subspace* $V_1 \subset V$ *which is invariant under all the* $\rho(g)$*.*

Example 2.9. *The representation* ρ_{perm} *of the symmetric group* S_k *introduced in (2.2.13), (2.2.14) is not irreducible, because* $e_1 + e_2 + \ldots + e_k$ *is clearly invariant under all permutation matrices. Its orthogonal complement is also invariant (because all permutation matrices are orthogonal). We have shown, then, that this representation can be written as a direct sum of a dimension 1 representation (the trivial representation) and of a dimension* $k-1$ *representation.*

Before finishing this section, let me notice that:

Lemma 2.1. *Each element in* S_k *can be represented as a composition of cycles of length 2. (Call this the "2-cycle decomposition")*

For example, $(13)(264) = (13) \circ (26) \circ (24)$ (**exercise**).

The lemma does not say that the 2-cycle decomposition is unique. There will be a minimal number of 2-cycles $n_{\min}(g)$ for a given element g. However, for any other 2-cycle decomposition, the number of cycles $n(g)$ will be odd if $n_{\min}(g)$ was odd, and even if $n_{\min}(g)$ was even. This motivates then the definition:

Definition 2.8. *The* ***parity*** $\pi(g)$ *of an element* $g \in S_k$ *is* $(-)^{n(g)}$*, where* $n(g)$ *is the number of 2-cycles in any 2-cycle decomposition of* g*.*

For example, the parity of (123) is $+1$. The parity of $(13)(264)$ is $(-1)^3 = -1$ (since we found earlier a 2-cycle decomposition with $n(g) = 3$).

A different (perhaps easier) way of defining parity in S_k is to consider the $\epsilon_{i_1 \ldots i_k}$ symbol; each of the indices goes from 1 to k. This tensor is characterized by being completely antisymmetric. The parity of the element is then given by

$$\pi \begin{pmatrix} 1 & 2 & \ldots & k \\ i_1 & i_2 & \ldots & i_k \end{pmatrix} = \epsilon_{i_1 i_2 \ldots i_k} . \tag{2.2.16}$$

For example, the parity of our old friend (2.2.9) is $\epsilon_{361254} = -1$, which confirms our older result. This is not really a different definition: when you compute this ϵ in your head, you are probably just using the 2-cycle decomposition!

We can also define

Definition 2.9. *The* alternating group A_k *is the subset of elements of* S_k *whose parity is positive.*

This subset is a group with the same composition law we had defined in S_k. The appropriate concept here is "subgroup", which we now proceed to define.

2.3 Subgroups; (semi)direct products

Let us now look "inside" a group.

Definition 2.10. *A subset H of a group G is a* ***subgroup*** *(and we will write $H < G$) if it is a group under the composition $\circ$ of G.*

Example 2.10.

- *The group $\mathbb{Z}_4$, whose elements are $\{e, r, r^2, r^3\}$, contains a subgroup $\{e, r^2\}$, which is isomorphic to $\mathbb{Z}_2$. (Exercise 1 at the end of these lecture notes will ask you to generalize this.)*
- *The group D_k (introduced in example 2.3) contains a subgroup isomorphic to $\mathbb{Z}_k$: its elements are r^i, $i = 1, \ldots, k$.*
- *The group D_k also contains a subgroup isomorphic to $\mathbb{Z}_2$: its elements are $\{1, \sigma\}$.*
- *The group S_k (introduced in example 2.8) contains a subgroup isomorphic to $\mathbb{Z}_k$: we can simply notice that $\begin{pmatrix} 1 & 2 & \ldots & k-1 & k \\ 2 & 3 & \ldots & k & 1 \end{pmatrix} = (12\ldots k)$ (we are using both notations for the group element; it is a single, long cycle) has the same properties as r, namely $r^k = e$, and $r^i \neq e$ $\forall i < k$.*

- *S_k also contains a D_k subgroup. We can think of this in the following way: D_k is the symmetry group of a regular polygon; identify then the vertices of this polygon with the elements of the set $\{1, 2, \ldots, k\}$, which get permuted by S_k. More formally: we already identified r in the previous paragraph; σ is then the element (using again the "cycle" notation) $\begin{pmatrix} 1 & 2 & \ldots & k-1 & k \\ k & k-1 & \ldots & 2 & 1 \end{pmatrix}$ — the "reflection". (In the "cycle" notation in (2.2.11), this is written $(1, k)(2, k-1)\ldots$.)*

- *By definition, the orthogonal group defined in example 2.4 is a subgroup of the group of invertible matrices: $\mathrm{O}(N, F) < \mathrm{Gl}(N, F)$.*

We have seen that S_k contains both $\mathbb{Z}_k$ and D_k. Actually, any finite group can be realized as a subgroup of S_k for a large enough k. (This is known as **Cayley's theorem**. The idea is to consider the elements of G as the set $\{12\ldots k\}$ permuted by S_k, and to associate to each $g \in G$ the permutation $g' \mapsto g'g$; for more details, see for example [1, Th. 2.F]). This theorem might sound disappointing: we have defined a group, and now we find that all finite groups can be found inside S_k! at least if we're interested in finite groups, perhaps we could have spared ourselves the abstract definition and work with S_k all the time. But actually, the abstract definition is usually much more powerful than using subgroups of S_k.

We will actually see other theorems of this type: an abstract definition that can be realized "within" a much more prosaic example. In this course, this will happen with manifolds (Whitney's theorem, 4.1) and with Lie algebras (Ado's theorem, 4.2).

Let me remark, by the way, that a representation of a group G induces a representation of a subgroup $H < G$: we can simply restrict ρ to H.

Example 2.11. *The representation of D_3 given in example 2.6 induces a representation of its subgroup $\mathbb{Z}_3 < D_3$: we can simply take $\rho(r)$ given there. Actually, if we identify $\mathbb{C} \equiv \mathbb{R}^2$, this representation is a particular case of the one given for $\mathbb{Z}_k$ in (2.2.6).*

Definition 2.11. *The right quotient G/H (or simply* ***quotient****) of a group G by a subgroup H is defined as the set of equivalence classes*

$$g_1 \cong g_2 \text{ if } \exists h \in H \text{ such that } g_1 = g_2 h \ . \tag{2.3.1}$$

We denote the equivalence class of g in G/H by gH.

A way to understand the notation gH is this: if we think of H as the set of its elements $H = \{e, h_1, h_2, \ldots\}$, then $gH = \{g, gh_1, gh_2, \ldots\}$; this is indeed an equivalence class, because all of these elements are equivalent with respect to $\cong$.

Definition 2.12. *A subgroup N of a group G is said to be* ***normal*** *(we will write $N \lhd G$) if*

$$gNg^{-1} = N \ , \qquad \forall g \in G \ ; \tag{2.3.2}$$

more explicitly, this means that $gng^{-1} \in N$ $\forall g \in G$, $\forall n \in N$. (One can actually prove that it is enough to check $gNg^{-1} \subset N$.)

Intuitively, a normal subgroup has a more "intrinsic" definition, in a sense that will hopefully become clear in the examples below.

The main reason to define a normal subgroup is that we have:

Theorem 2.2. *If N is a normal subgroup of a group G, $N \lhd G$, then the quotient G/N is a group.*

Proof. We need to define a composition of g_1N with g_2N. This is easily done by using the composition in G:

$$g_1Ng_2N = g_1g_2Ng_2^{-1}g_2N = g_1g_2N \ . \tag{2.3.3}$$

Associativity is easily checked; the identity element is simply the equivalence class $eN = N$. The inverse of an element gN is simply $g^{-1}N$, where g^{-1} is the inverse of g in G. $\square$

We can think of the quotienting procedure as simply a way to identify a subgroup with the identity.

Example 2.12.

- *The subgroup $\mathbb{Z}_2 < D_k$ defined in example 2.10 (whose elements are e and σ) is not normal: from the relations in the usual presentation we have $r^{-1}\sigma r = r^{-2}\sigma$, which does not belong to the subgroup.*

  Let's see what the fact that $\mathbb{Z}_2$ is not normal has to do with the quotient $D_k/\mathbb{Z}_2$. The point is that we would identify for example $r^i \cong r^i\sigma$. But then the original product law doesn't make sense in the quotient. For example:

  $$r^i r^j = r^{i+j} \ , \qquad (r^i\sigma)(r^j\sigma) = r^{i-j} \ ; \tag{2.3.4}$$

  but then we don't know the product of the equivalence class $r^i\mathbb{Z}_2 = \{r^i, r^i\sigma\}$ and of $r^j\mathbb{Z}_2 = \{r^j, r^j\sigma\}$; is it r^{i+j}, or r^{i-j}? the original composition is no longer well-defined on $D_k/\mathbb{Z}_2$. (Of course, this quotient is a set with k elements, and we might throw away the original composition and define another one that turns it into $\mathbb{Z}_k$, but that's cheating!)

- *The subgroup* $\mathbb{Z}_k < D_k$ *defined in example 2.10 (whose elements are* r^i, $i \in \mathbb{Z}$*) is normal: using the group relations given in example 2.3, it is easy to check that* gr^ig^{-1} *is of the form* r^k *for some* k *[exercise]. Since we are identifying all elements of the form* r^i *with the identity, all that remains is the element* σ*, and the identity.*

  In this case the problem we encountered in the previous point doesn't arise. The equivalence relation says $\sigma \cong \sigma r^k$*, for all* k*. But* $\sigma r^i \sigma r^j = \sigma^2 r^{i-j} = \sigma^2 = e$*, irrespectively of the representative* σr^i *that we choose in the equivalence class* $\sigma\mathbb{Z}_k = \sigma\{e, r, \ldots r^{k-1}\}$*. The quotient* $D_k/\mathbb{Z}_k$ *is then the group* $\mathbb{Z}_2$*.*

- *Intuitively, the difference between the two examples we just saw is the following:* $\mathbb{Z}_k$ *is the subgroup of all rotations, while* $\mathbb{Z}_2$ *is the subgroup generated by one particular reflection. In the sense,* $\mathbb{Z}_k$ *is defined in a more "intrinsic" way.*

- *The subgroup* $\mathbb{Z}_k < S_k$ *defined in example 2.10 is not normal for* $k > 3$ *(for* $k = 3$*,* $S_3 = D_3$ *and we fall back to the previous example; for* $k = 2$*,* $S_2 = \mathbb{Z}_2$*). Let us consider for example the composition* $(12) \circ (12345) \circ (12) = (21345)$ *(as explained before (2.2.12)). Since* (12) *and* (21345) *are not powers of* (12345)*, they are not in the subgroup* $\mathbb{Z}_k < S_k$ *we are considering. So we have found* $g = (12)$ *such that* $g\mathbb{Z}_k g^{-1} \notin \mathbb{Z}_k$*. This shows that* $\mathbb{Z}_k < S_k$ *is not a normal subgroup.*

We have looked a groups inside other groups; let us now consider the opposite. How can we extend a group to a bigger group? a simple way is to combine it with another group. A rather trivial way to do so is the following:

Definition 2.13. *Given two groups* G *and* K*, their* direct product $G \times K$ *is the set of pairs* $\{(g, k),\ g \in G,\ k \in K\}$*, with a composition*

$$(g_1, k_1) \circ (g_2, k_2) \equiv (g_1 g_2, k_1 k_2) \ . \tag{2.3.5}$$

(We omitted the composition symbols in G *and* K*.)*

Clearly, the order of $G \times K$ is

$$\#(G \times K) = \#(G)\#(K) \ . \tag{2.3.6}$$

There are two subgroups $G \times \{e\} = \{(g, e), g \in G\}$ and $\{e\} \times K = \{(e, k), k \in K\}$. Notice that these subgroups commute:

$$(g, e) \circ (e, k) = (g, k) = (e, k) \circ (g, e) \ . \tag{2.3.7}$$

It follows that both subgroups are normal in $G\times K$; the quotients are simply $G\times K/(G\times \{e\}\cong K$ and $G\times K/(\{e\}\times K)\cong G$.

We can use this observation to write a presentation for $G\times K$; this will provide a way to avoid the cumbersome pair notation in (2.3.5). Let $S_1=\{g_i\}$ be the generators for G, and $S_2=\{k_j\}$ be the generators for K. The generators for $G\times K$ will just be the union of all of them, $S=S_1\cup S_2$. In the "pairs" notation used in (2.3.5), a generator for G would now be written as (g_i,k_j). It is more practical, however, to write g_i instead of (g_i,e), and k_j instead of (e,k_j). Using the observation (2.3.7), then, we can write g_ik_j instead of (g_i,k_j). Actually, (2.3.7) tells us that it also makes sense to write k_jg_i for the same element. This means that we should impose the equivalence relations

$$gk\cong kg \text{ in } G\times K\ , \qquad \forall g\in S_1, K\in S_2\ . \tag{2.3.8}$$

Of course we should also remember whatever relations R_1 and R_2 we had in G and K. **To summarize:** we have a presentation

$$G\times K=\left\langle S_1\cup S_2\middle| R_1\cup R_2\cup\{gkg^{-1}k^{-1},\forall s_1\in S_1,s_2\in S_2\}\right\rangle\ . \tag{2.3.9}$$

This might look complicated, but it simply means that we consider words made up with the generators for G and for K, and that we add the relation $gkg^{-1}k^{-1}=e$, which is the same as imposing (2.3.8).

Example 2.13. *Let p and q are* coprime *integers (which means that* $\mathrm{gcf}(p,q)=1$*: they don't have common factors). Then $\mathbb{Z}_{pq}$ is isomorphic to $\mathbb{Z}_p\times\mathbb{Z}_q$. This is known as the **Chinese Remainder Theorem** (because it was found by Chinese mathematician Sun Zi around 400CE).*

There is a more interesting way to define a group from two groups G and N. Recall that an automorphism is an isomorphism of a group into itself. Call $\mathrm{Aut}(G)$ the set of automorphisms of G. [Exercise: this is a group!]

Definition 2.14. *Given two groups G and N, their **semidirect product** $G\ltimes N$ (or $N\rtimes G$; notice how the notation is asymmetric) is defined by the following presentation. Take as generators all the generators of G and N, just like one would do for $G\times N$. As relations, take all relations in G and N, but add the relation:*

$$g^{-1}ng\cong\phi_g(n)\ , \tag{2.3.10}$$

where ϕ_g is an automorphism of N that depends on g.[4]

[4] ϕ_g should be a homomorphism from G to $\mathrm{Aut}(N)$.

Just like for the direct product (2.3.9), we can summarize this construction by writing

$$G \ltimes N = \left\langle S_1 \cup S_2 \middle| R_1 \cup R_2 \cup \{g\phi_g(n)g^{-1}n^{-1}, \forall s_1 \in S_1, s_2 \in S_2\} \right\rangle . \tag{2.3.11}$$

Again, this might look complicated, but it simply means that we consider words made up with the generators for G and for N, and that we add to their generators the relation $g\phi_g(n)g^{-1}n^{-1} = e$, which is the same as imposing (2.3.10).

A few remarks about the concept of semidirect product:

- Just like in the direct product case, we have $\#(G \ltimes N) = \#(G)\#(N)$.
- If ϕ_g is the identity, namely $\phi_g(n) = n \ \forall n$, the semidirect product reduces to the direct product.
- N is a normal subgroup: $N \triangleleft (G \ltimes N)$. (This is why we called it N to begin with.) To see this, let us check that both the generators of G and of N satisfy $gNg^{-1} = N$. For a generator g_i of G, we have $g_i N g_i^{-1} = \phi_{g_i}(N) = N$ (because ϕ is an automorphism). For a generator n_j of N, we have $n_i N n_i^{-1} = N$, simply because $n_i \in N$.
- G is a subgroup of $G \ltimes N$, but in general is *not* a normal subgroup.
- The two previous observations explain the symbol $\ltimes$. You can think of it as $\triangleright$ (which is a triangle with a vertex pointing at N, which is a normal subgroup) next to $<$ (whose vertex points at G, which is a subgroup, but not necessarily a normal one).

Example 2.14.

- *The dihedral group D_k is a semidirect product:*

$$D_k \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_k . \tag{2.3.12}$$

To see this, we can simply recall the relations (2.2.2) of D_k, and recognize that $\sigma^2 \cong e$ is the relation for $\mathbb{Z}_2$, $r^k \cong e$ is the relation for $\mathbb{Z}_k$, and $\sigma r \sigma \cong r^{-1}$ is nothing but (2.3.10), where $\phi_\sigma(r) = r^{-1}$ is the automorphism of $\mathbb{Z}_k$. Notice that, as we noticed in example 2.10, $\mathbb{Z}_k$ is a normal subgroup of D_k, but $\mathbb{Z}_2$ is not, in agreement with our general remark above.

- *We have already considered the group $\mathrm{O}(N)$ of rotations (and reflections) in $\mathbb{R}^N$. We can consider a bigger group, which includes translations as well as rotations and reflections: in formulas, these are the transformations that send a $v \in \mathbb{R}^N$ to $E_{O,x}(v) = Ov + x$, where $O \in \mathrm{O}(N)$ is a rotation and $x \in \mathbb{R}^N$ is a vector that represents a translation. This is called the Euclidean group in N dimensions, and it is denoted by $\mathrm{E}(N)$ or by $\mathrm{ISO}(N)$.*

Let us work out the composition between two elements of $E(N)$. We know already how to multiply two rotations. We also know how to multiply translations: a translation $E_{1,x}$ by a vector $x \in \mathbb{R}^N$ acts simply as $v \mapsto v + x$. The subgroup of all translations is abelian, and it is in fact isomorphic to $\mathbb{R}^N$.

We are left with understanding how rotations and translations are multiplied. They do not commute: indeed

$$E_{O_1,x_1} E_{O_2,x_2} v = E_{O_1,x_1}(O_2 v + x_2) = O_1(O_2 v + x_2) + x_1 = (O_1 O_2) v + (O_1 x_2 + x_1) \ . \tag{2.3.13}$$

From this we see that

$$E_{O,0} E_{1,x} E_{O^{-1},0} = E_{1,Ox} \ . \tag{2.3.14}$$

We can be a bit more informal and denote a pure rotation $E_{O,0}$ directly by O; call a pure translation $E_{1,x} \equiv T_x$. Then (2.3.14) reads

$$O T_v O^{-1} = T_{Ov} \tag{2.3.15}$$

which is clearly of the form (2.3.10), with $\phi_O(x) = Ox$. This makes geometrical sense: rotating, translating, and rotating back, is equivalent to translating in a different direction [exercise: visualize this!] Summarizing, we have

$$E(N) \cong \mathrm{O}(N) \ltimes \mathbb{R}^N \ , \tag{2.3.16}$$

where $\mathbb{R}^N$ is the subgroup of translations T_x.

One way of remembering which one of the two factors in (2.3.16) is the normal subgroup is once again to ask which of the two is more "intrinsic" (remember the comment in the third point of example 2.12). $\mathbb{R}^N$ is the subgroup of all translations, while $\mathrm{O}(N)$ is the subgroup of some particular rotations, namely the ones about the origin. So $\mathbb{R}^N$ is more intrinsically defined, and it is the normal subgroup, while $\mathrm{O}(N)$ is not normal.

Exercise: show that

$$\mathrm{O}(N) \cong \mathrm{SO}(N) \rtimes \mathbb{Z}_2 \ . \tag{2.3.17}$$

The concept of semidirect product is some sort of inverse to the concept of taking the quotient by a normal subgroup:

$$(G \ltimes N)/N = G \ . \tag{2.3.18}$$

Because of this, it makes sense to study "simple" groups:

Definition 2.15. *A group is called* ***simple*** *if it has no normal subgroups.*

One can then "assemble" non-simple groups from simple ones by using semidirect products.

Finite simple groups have been *classified*! this means that we know exactly all the simple groups that can possibly exist. I will not attempt to reproduce the list here, but I can make a few remarks:

- Some steps of the classification have been achieved by brute force — provided by computers. The resulting proof is more or less unreadable (in its entirety) by humans.

- Most groups appear in infinite "families"; for example, one is the family of $\mathbb{Z}_p$, with p prime. ($\mathbb{Z}_k$, with k not prime, has many subgroups, which are all trivially normal because the group is abelian.) Another is the family of alternating groups A_k, defined in 2.9. (The symmetric group S_k is not simple, because $A_k \triangleleft S_k$.)

- There are then some "exceptional" finite simple groups; many are subgroups of a group called the **monster group**, presumably because of its cardinality:

$$\#(\text{monster group}) = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \sim 8\cdot 10^{53}\,. \tag{2.3.19}$$

2.4 Fun

We now quote some examples of groups that you may have encountered in your daily life, perhaps without recognizing them.

Notice first of all that the dihedral group D_k can be thought of as a subgroup of O(2): $D_k < \mathrm{O}(2)$ (as a consequence, also $\mathbb{Z}_k < \mathrm{O}(2)$). We could now try to investigate the finite subgroups of O(3). For simplicity, however, we will look at the subgroups of SO(3), which are simpler.

Theorem 2.3. *The finite subgroups of SO(3) (called* rotation groups*) are one of the following:*

- *The cyclic group $\mathbb{Z}_k$;*

- *The dihedral group D_k;*

- *T, the rotation group of the tetrahedron.*

- *O, the rotation group of the octahedron; this is also the symmetry group of the cube.*
- *I, the rotation group of the icosahedron; this is also the symmetry group of the dodecahedron.*

(The "rotation group" of a solid is the group of all elements of SO(3) that leave the solid invariant; it does not include reflections, which would have $\det = -1$*.)*

See for example [2, Sec. 1.8].

Notice that we mentioned all five regular polyhedra (the "Platonic solids"). In case you never heard this, the icosahedron and the dodecahedron are "dual". The icosahedron has 20 faces, 34 edges, 12 vertices. Each face is a triangle; at each vertex, five edges meet. The dodecahedron has 20 vertices, 34 edges, 12 faces. At each vertex, three edges meet; each face is a pentagon. One is obtained from the other by swapping faces with vertices. One way to do this is to "cut" each vertex of the icosahedron to produce a pentagon. One will produce a solid with pentagons and hexagons, which looks exactly like a (european) football, or like the C_{60} buckminster-fullerene molecule (see figure 3)[5]. If one keeps enlarging the pentagons and shrinks the hexagons, one obtains a dodecahedron. This explains why I is the symmetry group of both solids. A similar duality relation exists between an octahedron and a cube. The tetrahedron is dual to itself.

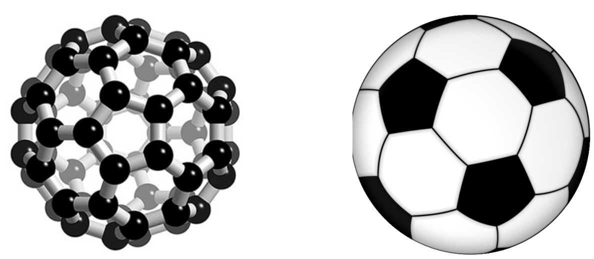


Figure 3: The C_{60} molecule, and a football.

Remark [ADE classifications; McKay correspondence]: In theorem 2.3 we saw two infinite series of groups ($\mathbb{Z}_n$, D_n) and three exceptional cases (T, O, I). There is

[5] As you can see, each carbon atom is bound to three other carbon atoms, which means one of its bonds must be double. So actually we would have to double some of the bonds in figure 3. There are many such "doublings"; each of them breaks some of the symmetries in the icosahedral group. It's probably not clear which of these so-called "Kekulé structures" is energetically favored. The problem should be familiar to you from the case of the benzene ring, where there are two such assignments; famously, in that case neither of the two dominates, but rather the ring is in a quantum superposition of the two.

a funky correspondence between these groups and some of the diagrams in figure 12, namely the two infinite series A_n, D_n and the three E_6, E_7, E_8. Those diagrams classify something completely different, as we will see, but there is a way in which the *same diagrams* can be associated to the finite groups we are studying here. This is called the McKay correspondence[6]; it is a mathematical mystery. (It has found an explanation of sorts in string theory.)

Let us go back to two dimensions. One can consider subgroups of the Euclidean group E(2) which contain at least two linearly independent translations. These are called *wallpaper groups*, because wallpapers have often such symmetry groups.

Theorem 2.4. *There are exactly 17 different wallpaper groups.*

Apparently, 13 of these 17 are used in the decorations of the Alhambra building in Granada. We see an example in figure 4.

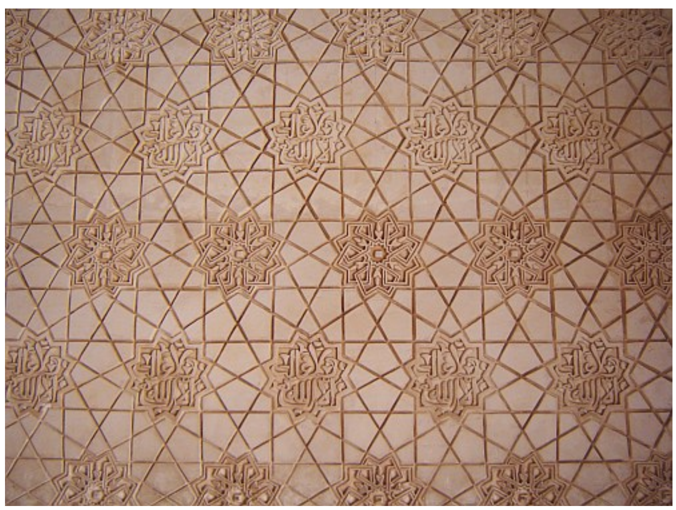


Figure 4: One of the patterns used in the Alhambra. It realizes one of the 17 “wallpaper groups”.

The three-dimensional analogues of wallpaper groups are called *space groups*. These are also classified: there are 230 of them. This classification is important for studying crystals, as you can probably imagine.

[6]Actually, the original correspondence is formulated not using the subgroups of SO(3), but those of $\mathrm{Sl}(2,\mathbb{C})$, or of SU(2); but subgroups of SU(2) are in one-to-one correspondence with subgroups of SO(3), because $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, as we will see.

A three dimensional object that many of you might have seen is **Rubik's Cube**. The group of symmetries of Rubik's Cube is, I hear:

$$(\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2) \; ; \tag{2.4.1}$$

recall that the alternating group A_k was introduced in 2.9. This is the group of moves of the cube; solving the cube means finding a group element that sends a given configuration into the one with all colors grouped together on each face.

3 Finite groups and their representations

Our aim in this section will be to classify the **complex irreducible representations** of finite groups. In this section, our group will be **finite** unless otherwise stated. Moreover, all representations will be understood to be complex: this means that the vector space on which G is represented is $\mathbb{C}^N$ for some dimension N. Recall also that a representation $\rho : G \times V \to V$ is "irreducible" if there are no subspaces of V (other than $\underline{0}$, and V itself) invariant under all $\rho(g)$, $\forall g \in G$ (see definition 2.7).

Lemma 3.1. ***[Schur's]****. Let $\rho_1 : G \times V_1 \to V_1$ and $\rho_2 : G \times V_2 \to V_2$ be two irreducible representations of G. Let $T : V_2 \to V_1$ be a linear map (a matrix), such that*

$$\rho_1(g)T = T\rho_2(g) \; , \qquad \forall g \in G \; . \tag{3.0.2}$$

Then:

i) If $V_1 \neq V_2$, $T = 0$.

ii) If $\rho_1 = \rho_2 = \rho$ (and hence $V_1 = V_2 \equiv V$), T is proportional to the identity on V.

Proof. The idea is simple: T cannot treat a subspace of V_1 or V_2 differently from the rest, or else it would make either ρ_1 or ρ_2 decomposable.

Let's see this more precisely. First, $\ker(T)$ is an invariant subspace for the representation ρ_2. Indeed: if $v \in \ker(T)$, $Tv = 0$; but then $0 = \rho_1 T v = T\rho_2 v$, which means that $\rho_2 v \in \ker(T)$. By assumption, the only invariant subspaces for ρ_2 should be V_2 or $\{\underline{0}\}$, so $\ker(T)$ is either the entire V_2 — in which case $T = 0$, which implies what we wanted to show — or $\{\underline{0}\}$. From now on we will assume $\ker(T) = \{\underline{0}\}$.

We now also have that $\mathrm{im}(T)$ is an invariant subspace for the representation ρ_1. Indeed, if $v_1 \in \mathrm{im}(T)$, it means that $\exists w_2$ such that $v_1 = Tw_2$. Applying ρ_1 to this, we see that $\rho_1 v_1 = \rho_1 T w_2 = T\rho_2 w_2$. So $\rho_1 v_1$ is also in $\mathrm{im}(T)$.

But once again, since $\mathrm{im}(T)$ is an invariant subspace for ρ_1, it can only be $\{\underline{0}\}$ — and once again, in this case $T = 0$ and we are done — or the entire V_1. But T has no kernel, so it must be an invertible square matrix, and $V_1 = V_2$.

We have now proven i). To prove ii), we repeat the argument using this time the operator $T - c1$, where c is a constant and 1 is the identity on V. For any c, $\ker(T - c1)$ is invariant for ρ, so it can only be the entire V or $\{\underline{0}\}$. It follows that it can only be proportional to the identity. □

3.1 Unitary representations

Definition 3.1. *A representation ρ is called* ***unitary*** *(recall: (2.2.4)) if $\rho(g)$ is a unitary matrix for all $g \in G$.*

The following theorem allows us to reduce us to unitary representations:

Theorem 3.2. *Any representation of a finite group is equivalent to a unitary representation.*

Proof. Let us call, as usual, ρ the representation, and V the vector space it acts on. Define now

$$M = \sum_{g\in G} \rho^\dagger(g)\rho(g) \ . \tag{3.1.1}$$

(Here and in what follows, $\rho^\dagger(g) \equiv (\rho(g))^\dagger$.) The important point about M is that it is invariant by the action of any $g \in G$:

$$\begin{aligned} \rho(g)^\dagger M \rho(g) &= \rho^\dagger(g)\Big(\sum_{g'\in G}\rho^\dagger(g')\rho(g')\Big)\rho(g) = \sum_{g'\in G}(\rho(g')\rho(g))^\dagger(\rho(g')\rho(g)) = \\ &= \sum_{g'\in G}\rho^\dagger(g'g)\rho(g'g) = \sum_{g''\in G}\rho^\dagger(g'')\rho(g'') = M \ . \end{aligned} \tag{3.1.2}$$

So if we can write $M = S^2$, we are done, because then (as you can check) $S\rho(g)S^{-1}$ is unitary.

Let us now show that M can be written as S^2. Clearly M is hermitian, so its eigenvalues m_i are real. They are also positive: for any $v \in V$,

$$v^\dagger M v = \sum_{g\in G} v^\dagger \rho^\dagger(g)\rho(g) v = \sum_{g\in G} ||\rho(g)v||^2 \geq ||\rho(e)v||^2 > 0 \ . \tag{3.1.3}$$

It follows that M can be written as $M = S^2$ for some matrix S. (Diagonalize $M = U^\dagger m U$, where $m = \mathrm{diag}(m_1, \ldots, m_N)$, and define $S = U^\dagger s U$, where $s = \mathrm{diag}(\sqrt{m_1}, \ldots, \sqrt{m_N})$.) This completes the proof. □

This is not true for infinite groups: an easy example is the following representation of $\mathbb{R}$: $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

We also have:

Lemma 3.3. *Let ρ be a representation which is decomposable: namely, such that there is a nontrivial subspace $V' \subset V$ which is invariant under ρ. Then the representation is reducible (that is, it can be written as a direct sum in an appropriate basis).*

Proof. Thanks to theorem 3.2, we can restrict our attention to unitary representations. For those, we have

$$\rho(g^{-1}) = \rho(g)^\dagger \ , \qquad \forall g \in G \ . \tag{3.1.4}$$

Recall also that the *orthogonal complement* $(V')^\perp$ of V' is the space of vectors orthogonal to all of V': $(V')^\perp \equiv \{w | w^\dagger v', \forall v' \in V'\}$. I claim that $(V')^\perp$ is also an invariant subspace. To see this, take any $v' \in V'$ and compute:

$$(\rho(g)w)^\dagger v' = w^\dagger \rho(g)^\dagger v' = w^\dagger \rho(g)^\dagger v' = w^\dagger \rho(g^{-1}) v' = 0 \ . \tag{3.1.5}$$

(In the last step, we have used the fact that $\rho(g^{-1})v' \in V'$.) This just means that $\rho(g)w \in (V')^\perp$, which is what we wanted to show.

Now we can divide $V = V' \oplus (V')^\perp$. Since both subspaces are invariant, ρ will be block-diagonal in this basis. □

We end this section with a lemma that generalizes theorem 3.2.

Lemma 3.4. *Let ρ_1 and ρ_2 be two unitary representations (of dimensions N_1 and N_2, respectively), and A a linear map from V_2 to V_1 (namely, an $N_1 \times N_2$ matrix). Then*

$$T(A) \equiv \sum_{g \in G} \rho_1^\dagger(g) A \rho_2(g) \tag{3.1.6}$$

satisfies $\rho_1(g)T(A) = T(A)\rho_2(g)$, $\forall g \in G$. (It then follows from Schur's lemma that $T(A)$ is zero when $V_1 \neq V_2$, and proportional to the identity when $V_1 = V_2$.)

Proof. This is a computation very similar to (3.1.2):

$$\begin{aligned} \rho_1(g)T(A) &= \sum_{g' \in G} \rho_1^\dagger(g^{-1}) \rho_1^\dagger(g') A \rho_2(g') = \sum_{g' \in G} \rho_1^\dagger(g' g^{-1}) A \rho_2(g') \\ &= \sum_{g'' \in G} \rho_1^\dagger(g'') A \rho_2(g'' g) = \sum_{g'' \in G} \rho_1^\dagger(g'') A \rho_2(g'') \rho_2(g) = T(A)\rho_2(g) \ . \end{aligned} \tag{3.1.7}$$

□

3.2 Functions on a group

Definition 3.2. *Let us denote by $\mathbb{C}[G]$ the space of complex functions on G (that is, $f : G \to \mathbb{C}$). It is a vector space: the sum of two functions $f_1 + f_2$ is defined by $(f_1 + f_2)(g) \equiv f_1(g) + f_2(g)$. The dimension of this vector space is $\#(G)$; in fact, $\mathbb{C}[G] \cong \mathbb{C}^{\#(G)}$ (each $\mathbb{C}$ factor is the space of possible values on a single element of G).*

On $\mathbb{C}[G]$, we can introduce the inner product

$$(f_1, f_2) \equiv \frac{1}{\#(G)} \sum_{g \in G} \overline{f_1(g)} f_2(g) \ . \tag{3.2.1}$$

This definition should look familiar: it is directly inspired from the inner product $v^\dagger w = \sum_{i=1}^N \bar{v}_i w_i$ in $\mathbb{C}^N$, and from the inner product on functions on $\mathbb{R}$ — the one that makes $L^2(\mathbb{R})$ a Hilbert space. Indeed, it reduces to the inner product in $L^2(\mathbb{R})$ if we take $G = \mathbb{R}$. But recall that in this whole section we are taking G finite.

A particularly important function on G is

$$G \ni g \mapsto \rho_{1,ij}(g) \equiv (\rho_1(g))_{ij} \in \mathbb{C} \ . \tag{3.2.2}$$

There are many functions like this: for each representation of dimension N, we have N^2 of them. We now show that they satisfy a remarkable orthogonality property:

Theorem 3.5. *Given two irreducible unitary representations ρ_1 and ρ_2, we have*

$$(\rho_{1,ij}, \rho_{2,kl}) = 0 \ , \qquad \text{if } \rho_1 \ncong \rho_2 \ , \tag{3.2.3a}$$

and

$$(\rho_{1,ij}, \rho_{2,kl}) = \frac{1}{N} \delta_{ik} \delta_{jl} \ , \qquad \text{if } \rho_1 = \rho_2 \equiv \rho \ , \tag{3.2.3b}$$

where N is the dimension of ρ.

Proof. Let us apply lemma 3.4 to the particular matrix $A = E^{(i,k)}$ whose only non-zero entry is a 1 on the position i, k (which, I recall, is called an *elementary matrix*):

$$E^{(i,k)}_{jl} \equiv \delta^i_j \delta^k_l \ . \tag{3.2.4}$$

We can think of these as of a basis for the vector space of all possible $N_1 \times N_2$ matrices. We get, by making indices explicit in the matrix multiplications:

$$\begin{aligned} [T(E^{(i,k)})]_{jl} &= \sum_{g \in G} (\rho_1^\dagger(g))_{jr} \delta^i_r \delta^k_s (\rho_2(g))_{sl} = \sum_{g \in G} (\rho_1^\dagger(g))_{ji} \rho_2(g)_{kl} \\ &= \sum_{g \in G} \overline{\rho_1(g))_{ij}} \rho_2(g)_{kl} = \#(G) (\rho_{1,ij}, \rho_{2,kl}) \ . \end{aligned} \tag{3.2.5}$$

We can now use lemmas 3.4 and 3.1. When $\rho_1 \ncong \rho_2$, $T(A)$ has to be vanish for all A; this shows (3.2.3a). When $\rho_1 = \rho_2 \equiv \rho$, $T(A)$ is proportional to the identity:

$$[T(E^{(i,k)})]_{jl} = c^{(i,k)}\delta_{jl} \ . \tag{3.2.6}$$

To determine the proportionality constant $c^{(i,k)}$ (which, as the notation shows, can depend on i and k), we can just take the trace of both sides. The trace of the left hand side of (3.2.6) gives

$$[T(E^{(i,k)})]_{jj} = \sum_{g\in G}(\rho^\dagger(g))_{ji}(\rho(g))_{kj} = \sum_{g\in G}(\rho(g))_{kj}(\rho^\dagger(g))_{ji} = \sum_{g\in G}\delta_{ki} = \#(G)\delta_{ki} \ . \tag{3.2.7}$$

The trace of the right hand side of $c^{(i,k)}$ gives $Nc^{(i,k)}$. So we have $c^{(i,k)} = \#(G)\delta_{ik}/N$. Putting this back in (3.2.6) and comparing with (3.2.5), we obtain (3.2.3b). □

Let us introduce an index a that counts representations up to equivalence, so that we will have a list $\{\rho^a\}$ of irreducible representations. We can then rewrite (3.2.3) as

$$(\rho^a_{ij}, \rho^b_{kl}) = \frac{1}{N_a}\delta^{ab}\delta_{ik}\delta_{jl} \ , \tag{3.2.8}$$

where N_a is the dimension of the representation ρ^a. The orthogonality (3.2.8) relation shows that the ρ^a_{ij} are linearly independent. Since they are elements of the space $\mathbb{C}[G]$ of complex functions on G, which has dimension $\#(G)$, this already tells us that $\#(G)$ is at least the number of all ρ^a_{ij}, which is $\sum N_a^2$; the sum is over all the irreducible representations. We will show in theorem 3.6 that the ρ^a_{ij} are in fact a basis for $\mathbb{C}[G]$, so that $\#(G) = \sum N_a^2$.

In order to prove that theorem, we need a trick: we introduce a particular representation of G:

Definition 3.3. *Consider a basis e^g of vectors in the space $\mathbb{C}^{\#(G)}$ of complex functions on a finite group G.* [7] *Then the* ***regular representation*** *ρ^{R} of G is defined by*

$$\rho^{\mathrm{R}}(g)[e^{g'}] = e^{gg'} \tag{3.2.9}$$

We can also write ρ^{R} in index notation (again using elements of G as indices; see footnote 7). Then we have

$$\rho^{\mathrm{R}}_{g_1,g_2}(g_3) = \delta_{g_1,g_2g_3} \ . \tag{3.2.10}$$

(The right hand side is the δ in the usual sense: it gives 1 if $g_1 = g_2g_3$ in G, and zero otherwise.)

[7] We will use g as an index; there are indeed $\#(G)$ possible values for this index, which is the expected cardinality for a basis in $\mathbb{C}^{\#(G)}$. This requires a bit of mental gymnastics, but I find it a convenient notation.

Example 3.1. *Let us write down the regular representation for the group* $\mathbb{Z}_2 = \{e, r\}$*,* $r^2 = e$*. This group has dimension 2, so* ρ^{R} *acts on* $\mathbb{C}^2$*, and it consists of* 2×2 *matrices:*

$$\rho^{\mathrm{R}}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \rho^{\mathrm{R}}(r) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.2.11}$$

.

The regular representation is never irreducible; rather, it will be a direct sum of irreducible representations,

$$\rho^{\mathrm{R}} = \oplus_a \underbrace{\rho_a \oplus \ldots \oplus \rho_a}_{c_a^{\mathrm{R}} \text{ times}} = \sum c_a^{\mathrm{R}} \rho^a \ . \tag{3.2.12}$$

What we mean by this formula is that each irrep ρ^a will appear c_a^{R} times in the direct sum decomposition; the first expression is formally correct, the second more informal but perhaps more readable.

We can now prove, as promised:

Theorem 3.6. *The functions* ρ^a_{ij}*, where* a *runs through all irreducible representations, are a basis for the space* $\mathbb{C}[G]$ *of functions on* G*. In particular,*

$$\#(G) = \sum N_a^2 \ . \tag{3.2.13}$$

Proof. The orthogonality (3.2.8) relation shows that the ρ^a_{ij} are linearly independent. All we have to show, then, is that they span $\mathbb{C}[G]$.

Let then f be any function $\in \mathbb{C}[G]$. As a particular case of (3.2.10) we have $\rho^{\mathrm{R}}_{g'e}(g) = \delta_{g',g}$. Then we can write

$$f(g) = f_{g'} \delta_{g',g} = f_{g'} \rho^{\mathrm{R}}_{g'e}(g) \ , \tag{3.2.14}$$

where we have defined $f_{g'} \equiv f(g')$; we introduced this notation so as to be able to use the summed-indices convention over indices $g \in G$ too. Since ρ^{R} is a sum of irreducible representations, (3.2.12), we have written f as a linear combination of the ρ^a_{ij}:

$$f(g) = f_{aij} \rho^a_{ij}(g) \ . \tag{3.2.15}$$

(The coefficients f_{aij} are related to the $f_{g'}$ and to the change of basis one has to perform to write the regular representation ρ^{R} in a block-diagonal form, whose blocks would be the irreducible representations ρ^a.) □

[Since the regular representation has dimension $\#(G) = \sum_a c_a^{\mathrm{R}} N_a$, (3.2.13) suggests

$$c_a^{\mathrm{R}} = N_a \ . \tag{3.2.16}$$

One can show that this is indeed the case.]

3.3 Characters; number of irreducible representations

We now restrict our attention to a particular type of function.

Definition 3.4. *Two elements g_1 and g_2 in a group G are said to be **conjugated** to one another if $\exists g \in G$ such that*

$$g_1 = g g_2 g^{-1} \ . \tag{3.3.1}$$

The set of all group elements conjugated to a given $g_1 \in G$ is called the conjugacy class *of g_1.*

Definition 3.5. *A function f on G is called* central *if it is constant on a conjugacy class:*

$$f(g') = f(gg'g^{-1}) \ , \qquad \forall g, g' \in G \ . \tag{3.3.2}$$

The subspace $\subset \mathbb{C}[G]$ of all central functions will be denoted by $\mathbb{C}_0[G]$.

An important example of central function is:

Definition 3.6. *The **character** of a representation ρ is defined by*

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)) = (\rho(g))_{ii} \ , \qquad \forall g \in G \ . \tag{3.3.3}$$

A character is clearly central:

$$\chi_\rho(gg'g^{-1}) = \mathrm{Tr}(\rho(gg'g^{-1})) = \mathrm{Tr}(\rho(g)\rho(g')\rho(g)^{-1}) = \mathrm{Tr}(\rho(g)^{-1}\rho(g)\rho(g')) = \mathrm{Tr}(\rho(g')) = \chi_\rho(g') \ . \tag{3.3.4}$$

Since characters are functions on G, we can consider their inner products. By applying (3.2.8) we have

$$(\chi_a, \chi_b) = \delta_{ab} \ . \tag{3.3.5}$$

Example 3.2.

- *In an abelian group, each conjugacy class only contains one element, because $gg_2g^{-1} = g_2$. So there are $\#(G)$ conjugacy classes.*
- *The identity e is always in a conjugacy class by itself: no other element is conjugated to it.*
- *Let us consider D_3. One conjugacy class is $\{e\}$. Because of the relation $\sigma r \sigma = r^{-1}$, we have that r and $r^2 = r^{-1}$ are in the same class $\{r, r^{-1}\}$. We are left with elements of the form σr^i. These are all in the same class: $r\sigma r^{-1} = \sigma r^{-2} = \sigma r$ shows that σ and σr are conjugated; $r^2 \sigma r^{-2} = \sigma r^{-4} = \sigma r^2$ shows that σ and σr^2 are conjugated. Summing up: the conjugacy classes for D_3 are:*

$$\{e\} \ , \qquad \{r, r^2\} \ , \qquad \{\sigma, \sigma r, \sigma r^2\} \ . \tag{3.3.6}$$

Example 3.3. *Let us compute the character for the representation of D_3 introduced in (2.2.7). By taking the trace of that equation, we have*

$$\chi(e) = 2 \ , \qquad \chi(r) = -1 \ , \qquad \chi(\sigma) = 0 \ . \tag{3.3.7}$$

We have computed in (3.3.6) the conjugacy classes: since we know that χ_ρ is constant within a conjugacy class, we don't need to compute it on any other element.

Lemma 3.7. *Two representations with the same character are equivalent. (In other words: a representation is characterized by its character — which explains the name.)*

Proof. Any representation can be written as the direct sum of irreducible representations:

$$\rho = \oplus_a \underbrace{\rho_a \oplus \ldots \oplus \rho_a}_{c_a \text{ times}} = \sum_a c_a \rho_a \ , \tag{3.3.8}$$

which generalizes (3.2.12) to any representation. Given a character χ_ρ, from (3.3.5) it follows:

$$(\chi_a, \chi_\rho) = c_a \ . \tag{3.3.9}$$

This means that we can extract the c_a, and hence the representation ρ, from the character χ_ρ. □

We have shown in theorem 3.6 that the space $\mathbb{C}[G]$ of functions on G is spanned by the functions ρ^a_{ij}. We now prove:

Theorem 3.8. *The characters χ^a of the irreducible representations of a finite group G are a basis for the space of central functions $\mathbb{C}_0[G]$. In particular, the number of irreducible representations ρ^a of a finite group G is equal to the number of its conjugacy classes.*

Proof. Let us begin by applying (3.2.15) to $f(g_1^{-1} g g_1)$:

$$f(g_1^{-1} g g_1) = \sum_a f_{aij} \rho^a_{ik}(g_1^{-1}) \rho^a_{kl}(g) \rho^a_{lj}(g_1) = \sum_a f_{aij} \overline{\rho^a_{ki}(g_1)} \rho^a_{kl}(g) \rho^a_{lj}(g_1) \tag{3.3.10}$$

Now let us impose that $f(g)$ is central:

$$\begin{aligned} f(g) = f(g_1^{-1} g g_1) &= \frac{1}{\#(G)} \sum_{g_1 \in G} f(g_1^{-1} g g_1) = \frac{1}{\#(G)} \sum_{g_1 \in G} \sum_a f_{aij} \overline{\rho^a_{ki}(g_1)} \rho^a_{kl}(g) \rho^a_{lj}(g_1) \\ &= \sum_a \frac{1}{N_a} f_{aij} \delta_{kl} \delta_{ij} \rho^a_{kl}(g) = \sum_a \frac{1}{N_a} f_{aii} \chi^a(g) \ . \end{aligned} \tag{3.3.11}$$

This proves that any central $f(g)$ is a linear combination of χ^a.

The second statement then follows from lemma 3.7. □

3.4 Character table

Let us now choose a representative g^a in each conjugacy class; let k_a be the number of elements in each class. Notice that we are using the same index ^{a} as we used to label irreducible representations: this makes sense because the number of conjugacy classes is the same as the number of irreducible representations, as we just showed in theorem 3.8.

With this notation, (3.3.5) can now be written (recalling the definition of the inner product in (3.2.1))

$$(\chi_a, \chi_b) = \frac{1}{\#(G)} \sum_c k_c \bar{\chi}_a(g_c) \chi_b(g_c) = \delta_{ab} \ ; \tag{3.4.1}$$

we have used the fact that a central function is constant on each conjugacy class. An equivalent way to summarize this is by introducing:

Definition 3.7. *The **character matrix** of a finite group G is*

$$U^G_{ab} = \sqrt{\frac{k_a}{\#(G)}} \chi_b(g_a) \ . \tag{3.4.2}$$

U^G is square, again because of theorem 3.8, and (3.4.1) tells us that $\bar{U}^G_{ca} U^G_{cb} = \delta_{ab}$, or

$$(U^G)^\dagger U^G = 1 \ . \tag{3.4.3}$$

In other words:

Theorem 3.9. *The character matrix U^G is unitary.*

A unitary matrix also automatically satisfies $UU^\dagger = 1$; in indices, this means $U^G_{ac} \bar{U}^G_{bc} = \delta_{ab}$, which becomes a formula for summing over characters:

$$\sum_c \chi_c(g_a) \bar{\chi}_c(g_b) = \frac{\#(G)}{k_a} \delta_{ab} \ . \tag{3.4.4}$$

Let us perform a couple of cross-checks. First of all, let us consider a particular case of (3.4.1). A representation that exists in any group is the trivial representation, that sends all elements to 1. So let us try to take $\rho^a = \rho^b =$ the trivial representation; we get

$$\sum_c k_c = \#(G) \ . \tag{3.4.5}$$

This is certainly true (summing the number of elements in all conjugacy classes we obtain the number of elements in the entire group), but boring. For a nicer test, let us move on to (3.4.4). A conjugacy class that exists in any group is $\{e\}$ (as already remarked in example 3.2). So take $g_a = g_b = e$. For any representation ρ_a, $\chi_a(e) = N_a$, the dimension of the representation. So (3.4.4) in this case gives

$$\sum_c N_c^2 = \#(G) \tag{3.4.6}$$

which we had already found earlier in (3.2.13). This makes sense, because we used theorem 3.6 to prove theorem 3.8, and the latter to show (3.4.4).

Example 3.4. *Let us consider $\mathbb{Z}_k$. We know that each element is a conjugacy class (example 3.2). So there should also be k representations. From (3.2.13) we also see that all $N_a = 1$: all representations are one-dimensional. We have already seen one in example 2.6. It is easy to find the others:*

$$\rho_l(r) = \omega_k^l \ , \tag{3.4.7}$$

where, as in example 2.6, $\omega_k \equiv e^{2\pi i/k}$. Let us write the character matrix for $k = 3$. The columns correspond to the representations ρ_i, the rows to the elements r^j:

$$\sqrt{3}U^{\mathbb{Z}_3} = \left.\begin{array}{c|ccc} & \chi_0 & \chi_1 & \chi_2 \\ \hline \{e\} & 1 & 1 & 1 \\ \{r\} & 1 & \omega & \omega^2 \\ \{r^2\} & 1 & \omega^2 & \omega \end{array}\right| . \tag{3.4.8}$$

It is easy to check that this matrix is unitary.

Example 3.5. *Let us now consider D_3. We know from (3.3.6) that there are three conjugacy classes. Hence there should also be three irreducible representations. One is the trivial representation ρ_0. The other is the one given in (2.2.7), which has dimension 2: let's call it ρ_2. So we are missing one. From (3.2.13) we know that it should have dimension one, because:*

$$\#(D_3) = 6 = 1^2 + 1^2 + 2^2 \ . \tag{3.4.9}$$

Let us call the missing representation ρ_1.

We know already two columns of the character matrix U^{D3}: the characters of the trivial representation, $(1, 1, 1)^t$, and the characters of the two-dimensional representation ρ_2 in (2.2.7), which were (trivially) computed in (3.3.7), giving $(2, -1, 0)$. We can complete the matrix, writing the second column (relative to the one-dimensional representation ρ_1 we *don't know yet) by imposing unitarity. The result is*

$$U^{D_3} = \begin{array}{r|ccc|} & \chi_0 & \chi_1 & \chi_2 \\ \hline \{e\} & 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ \{r, r^2\} & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ \{\sigma, \sigma r, \sigma r^2\} & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{array} \,. \tag{3.4.10}$$

So from the second column we read that ρ_1 should have character $(1, 1, -1)^t$. (The square roots are because of the factors $\sqrt{k_a/\#(G)}$ in the definition (3.4.2)). This determines the representation:

$$\rho_1(r) = 1 \ , \qquad \rho_1(\sigma) = -1 \ . \tag{3.4.11}$$

In retrospect, this is a pretty easy representation; we could have found it without these methods, of course.

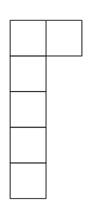
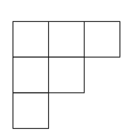
3.5 Extended example: representations of the symmetric group

The symmetric group S_k was introduced in example 2.8. In this section we will discuss its representation theory.

Let us first find its conjugacy classes. This might look like a formidable task, but let us look for example at (2.2.12), which shows that (12345) and (21345) are conjugated. If we had tried using any other 2-cycle instead of (12) in (2.2.12), it is easy to see that we would have found another 5-cycle (just like (12345) and (21345)). In fact, by composing 2-cycles, we can bring *any* 5-cycle in the form (12345). So the conjugacy class of (12345) contains all the five-cycles $(i_1\, i_2\, i_3\, i_4\, i_5)$.

We can now try with more complicated elements, for example $(13)(264) \in S_6$, that we used in (2.2.11) to illustrate the "cycle notation" for elements of S_k. We find that its conjugacy class contains all element with the number of cycles, of the same length: for example, another element in this conjugacy class would be (45)(136).

So we arrive at the conclusion that each conjugacy classes is characterized by the number of cycles, and by the length of each cycle. A visual notation for this is to use stacks of boxes: we use a vertical stack of k boxes to denote a cycle of length k. For example, the conjugacy class of $(12345) \in S_6$ and of $(13)(264) \in S_6$ would be denoted by

 and  (3.5.1)

respectively. In the first case, there are five rows and then one, because there is a five-cycle and a one-cycle (which, by convention, we didn't show when writing (12345)); in the second case, there is a 3-cycle, a 2-cycle, and a 1-cycle (which we didn't show when writing (13)(264), for the same reason).

Definition 3.8. *A **Young tableau** of order k is a way of stacking k squares in columns, ordered in such a way that the $(i+1)$-th column is not deeper than the i-th.*

The definition might sound obscure, but it's just a way of formalizing pictures such as in (3.5.1).

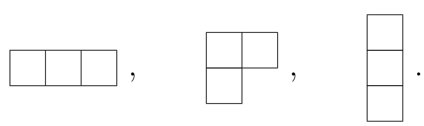
So the conjugacy classes for S_k are in one-to-one correspondence with Young tableaux of order k.

Example 3.6. *As a preamble to this exercise, let us first notice that $S_3 = D_3$. The isomorphism is given by*

$$(123) \cong r \ , \qquad (12) \cong \sigma \ . \tag{3.5.2}$$

The idea behind this isomorphism is that D_3 is the symmetry group of a triangle, and S_3 is the permutation group of 3 elements. In this case, all permutations of the vertices can be realized by rotations and reflections.

For S_3, there are three possible Young tableaux:



(3.5.3)

Since $S_3 \cong D_3$, these three tableaux should be in correspondence with the three conjugacy classes we have found for D_3 in (3.3.6). Let us check: the first conjugacy class correspond to three 1-cycles, which is just the identity. So this correspond to $\{e\}$. The second corresponds to one 2-cycle and one 1-cycle. The elements of this class are (12)*,* (23)*,* (31)*. Under the isomorphism (3.5.2), this becomes the class $\{\sigma, \sigma r, \sigma r^2\}$. Finally, the third diagram represents one 3-cycle. There are two elements like this:* (123) *and* (132)*. These correspond to $\{r, r^2\}$.*

It would now be an interesting exercise in combinatorics to count the Young diagrams of order k. That would give us the number of irreducible representations of S_k. We don't need to do that, however: we can just think that irreducible representations of S_k should also be in one-to-one correspondence with Young tableaux of order k.

How to do this is described in detail in [3, Ch.1.24], but here is the idea. Consider the vector space $\mathbb{C}^{k!}$. We denote the elements of a basis in this space by $|i_1 i_2 \dots i_k\rangle$. We will now associate to each Young tableau Y a *subspace* $V_Y \subset \mathbb{C}^{k!}$. Consider a way to fill a Young tableau, as in

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \; . \tag{3.5.4}$$

To this, associate the state obtained by first symmetrizing in each row, and then anti-symmetrizing in each column:

$$|123\rangle + |213\rangle - |321\rangle - |231\rangle \; . \tag{3.5.5}$$

There are five more ways of filling the Young tableau with numbers, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\ldots$; but they are not linearly independent, and their span has actually dimension 2. This span is defined to be $V_{\tiny\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}$, and corresponds to the representation of dimension 2 of D_3 that we saw in (2.2.7).

Moreover, if we associate a representation to $\begin{array}{|c|c|c|}\hline \, & \, & \, \\ \hline\end{array}$ and $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \, \\ \hline\end{array}$, we find that they both have dimension 1 (they correspond to the completely antisymmetric and completely antisymmetric state, respectively). So we conclude that S_3 has three irreducible representations, of dimension 1, 1, and 2. This is in agreement with our conclusion in example 3.5.

4 Lie groups and algebras

We now leave finite groups and start looking at infinite groups.

We have already seen that for example $\mathbb{Z}$ and $\mathbb{R}$ are infinite groups. The difference between these two examples is that the first is "discrete", the second is "continuous". We will not devote much attention to the first type, whereas we will study the second type a lot, starting from this section.

Unfortunately, we first have to formalize what we mean by "continuous". You might have already seen already some of the following geometrical definitions, but we're going to have to go through them anyway. In what follows, I will use the word **space** for a set with a Hausdorff topology.

4.1 Manifolds

Definition 4.1. *A* homeomorphism ϕ *between two spaces* A *and* B *is a continuous bijective map whose inverse* ϕ^{-1} *is also continuous.* A *and* B *are then said to be* homeomorphic.

Definition 4.2. *A* ***manifold*** *(*varietà*)* M *of dimension* $N \equiv \dim(M)$ *is a space which is locally homeomorphic to* $\mathbb{R}^N$*: namely, such that for every point* $p \in M$ *there exists an open neighborhood* U_p *of* p *which is homeomorphic to an open subset of* $\mathbb{R}^N$.

Example 4.1.

- $\mathbb{R}^N$ *is a manifold.*
- *The sphere*

$$S^N = \Big\{ \sum_{i=1}^{N+1} x_i^2 = 1 \Big\} \subset \mathbb{R}^{N+1} \tag{4.1.1}$$

is a smooth manifold. Let us see this explicitly, but for simplicity in the case $N = 2$*, so that* $S^2 = \{x^2 + y^2 + z^2\} \subset \mathbb{R}^3$*. Consider the two open sets:*

$$U_{\rm S} = S^2 - (0,0,1) \ , \qquad U_{\rm N} = S^2 - (0,0,-1) \ . \tag{4.1.2}$$

S *and* N *stand for South and North:* $U_{\rm S}$ *is the sphere without the "North Pole"* $(0,0,1)$*,* $U_{\rm N}$ *is the sphere without the "South Pole"* $(0,0,1)$*. The map* $\phi_{\rm S}$ *from* $U_{\rm S}$ *to* $\mathbb{R}^2$ *is the* stereographic projection *illustrated in figure 5.*

A similar map $\phi_{\rm N}$ *can be defined from* $U_{\rm N}$ *to* $\mathbb{R}^2$*. The Emblem of the United Nations results from applying* $\phi_{\rm N}$ *to a subset of the surface of the Earth.*

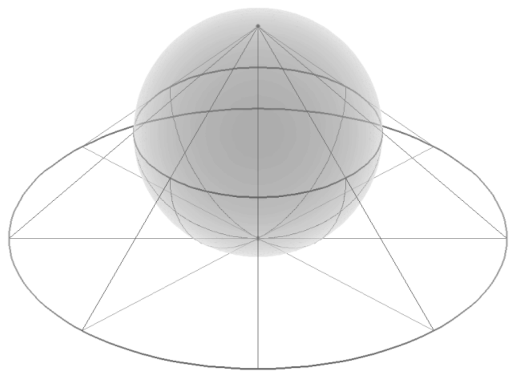


Figure 5: Stereographic projection.

Since any point in S^2 has either $U_{\rm S}$ or $U_{\rm S}$ (or both) as possible neighborhoods, and since $\phi_{\rm S}$ and $\phi_{\rm N}$ are homeomorphisms, S^2 is a manifold.

This discussion generalizes easily to S^N: again one can define two open sets, this time $U_{\rm S} = S^N - (0, 0, \ldots, 1)$, and $U_{\rm N} = S^N - (0, 0, \ldots, -1)$.

In the definition of manifold, no concept of "derivative" entered. For this reason, we don't know yet what it means to take derivatives of functions on a manifold. A possible way is to give coordinates on M. However, given a sufficiently complicated manifold, for example S^2, a coordinate system is not going to be "well-defined" everywhere. For example, on S^2, the coordinates (ϕ, θ) are not so nice near the two poles: what is ϕ at the North Pole? The remedy for this is to introduce coordinate systems which behave well in open subsets.

Definition 4.3. *Let M be a manifold of dimension N. A* chart (U, ϕ) *is an open set $U \subset M$ with a homeomorphism ϕ from U to a subset of $\mathbb{R}^N$. An* atlas *is a set of charts $\{(U_i, \phi_i)\}$ such that $\cup_i U_i = M$. The manifold M is said to be a* ***smooth manifold*** *(varietà liscia) if it has an atlas such that the* transition functions

$$g_{ij} \equiv \phi_j \circ \phi_i^{-1} \tag{4.1.3}$$

are C^∞ (whenever $U_i \cap U_j$ is non-empty; otherwise the composition doesn't exist).

Let us analyze a bit this definition. First of all, instead of considering homeomorphisms around every point, we have "economized" by covering the space with open sets U_i; each of these can serve as a neighborhood for all the points it contains.

Now, you can think of the ϕ_i as a way of assigning a coordinate system to each U_i: if you think of a coordinate x^m in $\mathbb{R}^N$ as a map from $\mathbb{R}^N$ to $\mathbb{R}$, then in the chart U_i we can also define a map

$$x^m_{(i)} : M \to \mathbb{R} \ , \qquad x^m_{(i)}(p) = x^m(\phi_i(p)) \ ; \tag{4.1.4}$$

and we can think of this map as of a coordinate on U_i.

On $U_i \cap U_j$ (when it's not empty), we have then two sets of coordinates, because we can apply both maps ϕ_i and ϕ_j. The transition functions g_{ij} in (4.1.3) relate the coordinate systems $x^m_{(i)}$ and $x^m_{(j)}$. Notice that g_{ij} is a map from a subset of $\mathbb{R}^N$ to another, so it makes sense to demand that it is C^∞.

Since we now have a coordinate system on M, we can decide when a function is differentiable:

Definition 4.4. *A function $f : M \to \mathbb{R}$ is C^∞ or* smooth *in $p \in M$ when, if (U, ϕ) is the chart that contains p, the function $f \circ \phi^{-1}$ is C^∞. The space of C^∞ functions on M will be denoted $C^\infty(M)$.*

Again here $\circ$ is the composition of maps (from right to left). The trick is that ϕ^{-1} takes us from a subset of $\mathbb{R}^n$ to M, and then f from M to $\mathbb{R}$; so $f \circ \phi^{-1}$ goes from a subset of $\mathbb{R}^N$ to $\mathbb{R}$, and we can decide whether it's differentiable or not. This just amounts to taking derivatives in the coordinate system x^m defined by the chart (U, ϕ).

Example 4.2.

- *$\mathbb{R}^N$ is a smooth manifold.*
- *S^N is a smooth manifold. The charts are just $U_{\rm S}$ and $U_{\rm N}$ given in equation (4.1.2). There is only one transition function, $g_{\rm SN} \equiv \phi_{\rm S} \circ \phi_{\rm N}^{-1}$; one can check easily that it is C^∞.*

A choice of atlas on a manifold is not unique. For example, we could have covered the sphere S^2 with another set of charts. We want to have a way of deciding when two differentiable manifolds "are the same":

Definition 4.5. *Let M_1 and M_2 be two smooth manifolds. A map f from M_1 to M_2 is defined to be C^∞, or* smooth*, at $p \in M_1$ if, calling (U, ϕ) the chart that contains $p \in M_1$ and (V, ψ) the chart that contains $f(p) \in M_2$, the map $\psi \circ f \circ \phi^{-1}$ is C^∞.*

A map $f : M_1 \to M_2$ is a ***diffeomorphism*** *if it is bijective, and if f and f^{-1} are both C^∞ everywhere.*

Again the idea is that $\psi \circ f \circ \phi^{-1}$ is a map from a subset of $\mathbb{R}^n$ to another, and so we can decide whether it is C^∞ by taking partial derivatives.

Remark: we have had to define a diffeomorphism after having defined a smooth manifold, whereas we defined a homeomorphism before defining a manifold. We couldn't have defined a diffeomorphism before defining a smooth manifold, because we would have had to decide what "differentiable" means! whereas we know what "continuous" means as soon as we have a topology.

Example 4.3. *The map $f(x) = x^3$ from $\mathbb{R}$ to itself is a homeomorphism (it is bijective, and its inverse $f^{-1}(y) = y^{1/3}$ is continuous), but it is not a diffeomorphism (f is C^∞, but f^{-1} isn't).*

Remark: The example we just saw was easy enough to find. It is much harder to find two homeomorphic smooth manifolds M_1 and M_2, which are not diffeomorphic. A given homeomorphism between M_1 and M_2 might not be a diffeomorphism, but that might just mean we haven't looked hard enough — and that a diffeomorphism does in fact exist. For example, let us consider $M_1 = S^N$, and M_2 homeomorphic to it. Is M_2 necessarily also diffeomorphic to it? Not necessarily: a smooth manifold which is homeomorphic to a sphere but not diffeomorphic to it is called "exotic sphere". There are no exotic spheres in dimension $N = 1, 2, 3, 5, 6$, but there are 27 exotic spheres in dimension $N = 7$. It is not known whether there are exotic S^4s. There are infinitely many smooth structures on $\mathbb{R}^4$, however.

Remark: We defined the sphere S^N (example 4.1) as a subspace of $\mathbb{R}^{N+1}$. The definition of manifold allows us to look at S^N "intrinsically", which means that we are only using coordinates on S^N itself, and never the coordinates of $\mathbb{R}^{N+1}$. (In the case of S^2, we can use the polar coordinates θ, ϕ rather than x^1, x^2, x^3 with one constraint.) Both points of view are possible in general: **Whitney's embedding theorem** says that we can always realize a smooth manifold of dimension M as a smooth subspace of $\mathbb{R}^{2N}$. In spite of this, the intrinsic approach is often more practical.

4.2 Vector fields and tangent space

A particularly interesting case of diffeomorphism is one from a manifold M to itself. On the one hand, this is just a map from M to itself. On the other hand, we can view it as a way of formalizing the idea of a "change of coordinates". Indeed, if we are working on a chart with coordinates x^m, a diffeomorphism will give functions

$$\tilde{x}^m(x^1, \ldots, x^N) \equiv \tilde{x}^m(x) \ , \qquad m = 1, \ldots, N \tag{4.2.1}$$

that we can also consider as a new set of coordinates on M. These "active" and "passive" points of view are familiar to you from Lorentz transformations or rotations (which are indeed particular cases of diffeomorphisms).

Consider a family ϕ_t of such diffeomorphism from M to itself, depending smoothly on a real parameter t, such that $\phi_0 = \mathrm{Id}$, the identity diffeomorphism. To fix ideas, you might want to think of ϕ_t as the time evolution in a configuration space. Now the *derivation* is D_ϕ defined as follows: it takes a function $f(p)$ on M, and it gives another function $D_\phi[f](p)$ defined by

$$D_\phi[f](p) \equiv \lim_{\delta t \to 0} \frac{f(\phi_{\delta t}(p)) - f(p)}{\delta t} \ . \tag{4.2.2}$$

This looks just like the definition of a derivative. In fact:

Example 4.4. *Let $M = \mathbb{R}$, and ϕ_t the family of diffeomorphisms given by translations: $\phi_t(x) = x + t$. Then we see that $D_\phi[f](x) \equiv \lim_{\delta t \to 0} \frac{f(x+\delta t) - f(x)}{\delta t} = f'(x)$.*

We can generalize this example. Any chart U will induce coordinates x^m (see (4.1.4)). Then the family of diffeomorphism $\phi_{t,m} \equiv x^n \to x^n + \delta^n{}_m t$ translates the coordinate x^m; the associated derivation defined in (4.2.2) reads in this case

$$D_{\phi_{t,m}} = \partial_m \equiv \frac{\partial}{\partial x^m} \ . \tag{4.2.3}$$

More generally, a diffeomorphism ϕ can be written, in a chart with coordinates x^m, by a transformation $x^m \to \tilde{x}^m(x)$. Infinitesimally,

$$x^m \to x^m + \delta x^m(x) \ . \tag{4.2.4}$$

We have written $\delta x^m(x)$ to emphasize that these components can depend on the point x. D_ϕ can then be written as a linear combination of the (4.2.3):

$$D = \delta x^m(x) \partial_m \ . \tag{4.2.5}$$

This is a directional derivative. The components δx tell us where the map is telling us to go, for very small t. If we think of ϕ_t as a time evolution, $\delta x(x)$ are the velocities at every point. We can imagine a collection of little "arrows", or "vectors", one for each point of M. This motivates the following name:

Definition 4.6. *A* ***vector field*** *v on M is a linear operator from the space of smooth functions $C^\infty(M)$ into itself, that on each chart is a directional derivative $v = v^m \partial_m$ (as in (4.2.5)).*

Notice that the definitions on two overlapping charts U and $\tilde{U}$ should agree, or else we don't know what the operator v is: in other words, v would not be well-defined. As in (4.1.4), U gives us coordinates x, and $\tilde{U}$ gives us coordinates $\tilde{x}$. Demanding that v should be well-defined gives us

$$v = v^m \frac{\partial}{\partial x^m} = \tilde{v}^n \frac{\partial}{\partial \tilde{x}^n} = \tilde{v}^n \frac{\partial x^m}{\partial \tilde{x}^n} \frac{\partial}{\partial x^m} \tag{4.2.6}$$

so that

$$v^m(x) = \frac{\partial x^m}{\partial \tilde{x}^n} \tilde{v}^n(\tilde{x}(x)) \ . \tag{4.2.7}$$

This is how a vector should transform under a change of coordinates from a chart to another, described by $\tilde{x}(x)$. Strictly speaking, this is not a diffeomorphism, since $\tilde{x}(x)$ is only defined on the intersection of the two charts, $U \cap \tilde{U}$. However, (4.2.7) is actually also valid for describing the transformation law of a vector field under a true diffeomorphism, defined everywhere on M.

If we have two vector fields v and w, we can then compute the action of their commutator:

$$[v, w]f = v^m \partial_m (w^n \partial_n f) - w^n \partial_n (v^m \partial_m f) = (v^m (\partial_m w^n) - w^m (\partial_m v^n)) \partial_n f \tag{4.2.8}$$

where we have cancelled one term because $\partial_m \partial_n f = \partial_n \partial_m f$.

Definition 4.7. *The **Lie bracket** of two vector fields v, w is their commutator as operators:*

$$[v, w] \equiv (v^m (\partial_m w^n) - w^m (\partial_m v^n)) \partial_n \ . \tag{4.2.9}$$

From a vector field v we can also reconstruct a family ϕ_t of diffeomorphisms. The idea is simply to 'follow the little arrows'.

Definition 4.8. *Given a vector field v, suppose there exists a solution $x(t)$ ($x(t) \in M$, $\forall t$) to the differential equation on M*[8]

$$\dot{x}^m(t) = v^m(x(t)) \ \forall t \ , \qquad x^m(0) = x_0^m \ , \tag{4.2.10}$$

where $x_0 \in M$, and x_0^m are its local coordinates. The exponential $\exp(tv)$ *of a vector field v is then the family of diffeomorphisms (depending on a real parameter t)* $\exp(tv)$ *that takes x_0 to the point $x(t)$ whose local coordinates are the $x^m(t)$:*

$$\exp(tv)[x_0] \equiv x(t) \ . \tag{4.2.11}$$

[8] We write this equation in local coordinates for clarity, but one can verify that the equation remains the same under changes of coordinates $\tilde{x}(x)$.

The curve described on M by varying t in $x(t)$ is called the ***integral curve*** *of the vector v.*

Remark: One can use this intuition about vector fields to give a geometrical meaning of the Lie bracket (4.2.9). If one applies $\exp[tv]$ and then $\exp[tw]$ to a point p, or first $\exp[tw]$ and then $\exp[tv]$, one usually doesn't find the same result. The difference in the two operations, when t is very small, is the Lie bracket $[v, w]$: see figure 6.

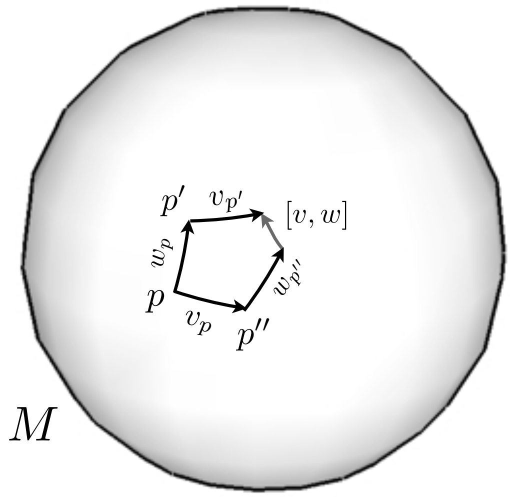


Figure 6: The geometrical meaning of the Lie bracket.

Let us however see what happens when the vector is constant in a given coordinate system:

Example 4.5. *The derivation in example 4.4, in the notation introduced in (4.2.3), reads $v = \frac{\partial}{\partial x} = \partial_x$. Since we are in one dimension, there is only one component, and it is $v^1 = 1$. The differential equation (4.2.10) in this case is*

$$\dot{x}(t) = 1 \ , \qquad x(0) = x_0 \ . \tag{4.2.12}$$

The solution is $x(t) = x_0 + t$. So

$$\exp(t\partial_x)[x_0] = x_0 + t \tag{4.2.13}$$

which is nothing but the family ϕ_t of diffeomorphisms we started from in example 4.4.

On a chart U, we can think of a vector field as of a map from U to $\mathbb{R}^N$. So we can imagine a copy of this $\mathbb{R}^N$ at every point p, where the v^m take values. This copy of $\mathbb{R}^N$ is called the *tangent space* T_pM at $p \in M$. We can formalize this in an alternative way. Let us consider a curve γ passing through $p \in M$. We can think of γ as a map

$$\gamma : \mathbb{R} \to M \ . \tag{4.2.14}$$

$\gamma(t)$ is the position on the curve at a "time" t.

Definition 4.9. *Let us consider the space of all the curves γ passing through a point $p \in M$ at time $t = 0$: $\Gamma_p \equiv \{\gamma : \mathbb{R} \to M | \gamma(0) = p\}$. On this space, we introduce the equivalence relation that says that $\gamma_1 \cong \gamma_2$ if their velocities at $t = 0$ are the same. In terms of the chart (U, ϕ) containing p, this means $\partial_t(\phi \circ \gamma_1)(0) = \partial_t(\phi \circ \gamma_2)(0)$. (We have to use ϕ to have some coordinates in which to measure the 'velocities'. Still more explicitly, in terms of the coordinates x^m in the chart U, we can write $\partial_t x^m(\gamma_1(0)) = \partial_t x^m(\gamma_2(0))$). The equivalence class*

$$T_pM = \{\gamma : \mathbb{R} \to M | \gamma(0) = p\}/\{\gamma_1 \cong \gamma_2 \text{ if } \partial_t(\phi \circ \gamma_1)(0) = \partial_t(\phi \circ \gamma_2)(0)\} \tag{4.2.15}$$

is called the ***tangent space*** *at a point $p \in M$. It is a vector space of dimension $N = \dim(M)$.*

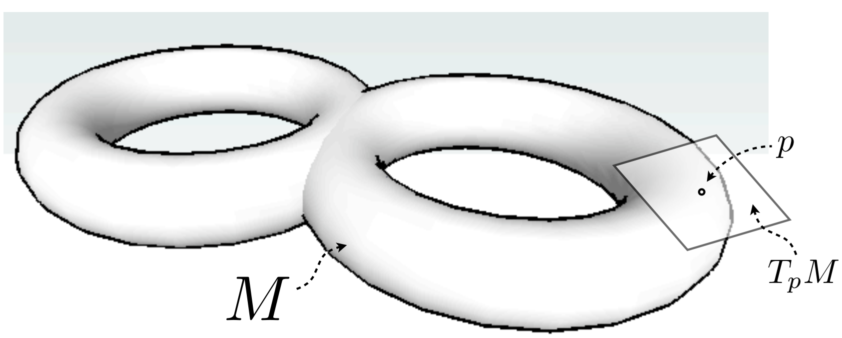


Figure 7: The tangent space T_pM at a point $p \in M$.

The idea behind (4.2.15), as messy as that definition might look like, is very simple: we are defining the space of all "directions" that one can decide to walk from $p \in M$ as the equivalence classes of all curves that start at p and that "begin" in the same way.

Another way of saying this is that a tangent space is a "linearization" of the manifold. In a sense, M looks like the vector space T_pM very close to M; see figure 7. To formalize this idea, one would like to find a map from T_pM to M, which is bijective at least in a neighborhood of p. If we were able to extend an element $v_p \in T_pM$ to a vector field v, we could then consider the diffeomorphism $\exp[tv]$ in (4.2.11), and apply this map to p. One would then have a map from T_pM to the manifold M. One possible idea would be to make v constant over M, and equal everywhere to the same value v_p. Unfortunately, the notion of "constant" vector doesn't really make sense! If we look at (4.2.7), we see that the v^m are constant in the coordinates x^m, the components $\tilde{v}^m$ in another set of coordinates $\tilde{x}^m$ are not constant. So for the time being we have to give up to this idea. You will see later (this will likely happen first in your General Relativity class) that there is actually a way to define what a constant vector is; more precisely, this will be called a "covariantly constant" vector. Then there will also be a notion of *exponential map* : $T_p \to M$, defined applying (4.2.11) to such covariantly constant vectors.

It would be tempting to define the tangent space T_pM just as a copy of $\mathbb{R}^N$ "attached" to p. The reason the abstract definition is better is that it is *intrinsic*: it doesn't depend on a choice of coordinates. The coordinates are given by an atlas, but it is nice to have the possibility of changing atlas. Also, on overlaps $U_i \cap U_j$, there are two competing coordinates, the $x^m_{(i)}$ and $x^m_{(j)}$ (see again (4.1.4)), and the components of a vector field v are not the same in the two coordinates, as (4.2.7) shows. This also implies that a vector field is **not** a function from M to $\mathbb{R}^N$.

A vector field v is a collection of little arrows, one for every point in M. The tangent space T_xM is the collection of all possible little arrows at a given point $x \in M$. The mental pictures are clearly similar, but let us tie these definitions together: can we "evaluate" a vector field v at a point x? the answer should be yes, but it is fun to see why using the abstract definitions above. Given a vector field v and a point $x \in M$, we apply the family of maps $\exp(tv)$ defined in (4.2.11) to x. This is the integral curve defined in definition 4.8. The equivalence class of this curve (in (4.2.15)) is v_x, the "evaluation" of the vector v at the point x.

4.3 Lie groups

We are now ready for the most important concept in this course:

Definition 4.10. *A **Lie group** G is a smooth manifold which is also a group, such that its composition $\circ : G \times G \to G$ and the inverse $(g \mapsto g^{-1}) : G \to G$ are smooth (see definition 4.5). The dimension of G is its dimension as a manifold.*

Example 4.6.

- *The group $\mathrm{Gl}(N, \mathbb{R})$ of invertible matrices is a Lie group. Since the* *condition* $\det(M) \neq 0$ *is not a constraint but an inequality,* $\mathrm{Gl}(N, \mathbb{R})$ *has dimension* N^2*. The subgroup*

$$\mathrm{Sl}(N) \equiv \{M \in \mathrm{Gl}(N) | \det(M) = 1\} \tag{4.3.1}$$

is a Lie group of dimension $N^2 - 1$*.*

- *The group* $\mathrm{O}(N)$ *introduced in example 2.4 is a Lie group. It has two components: the subgroup* $\mathrm{SO}(N)$ *of matrices with determinant* $= 1$*, and the subset (not a subgroup) of matrices with determinant* $= -1$*. To compute the dimension of* $\mathrm{SO}(N)$ *as a manifold, consider the group* $\mathrm{Gl}(N, \mathbb{R})$ *of all matrices, and subtract the number of all constraints coming from* $OO^t = 1$*. This just says that the columns of* O *are orthonormal vectors in* $\mathbb{R}^N$*: these are* $N(N+1)/2$ *independent constraints. Subtracting this from* $\dim(\mathrm{Gl}(N, \mathbb{R})) = N^2$*, we get*

$$\dim(\mathrm{SO}(N)) = \frac{1}{2} N(N-1) \ . \tag{4.3.2}$$

- *The group* $\mathrm{U}(N)$ *introduced in example 2.5 is a Lie group.*

- *The subgroup* $\mathrm{SU}(N) < \mathrm{U}(N)$ *defined by*

$$\mathrm{SU}(N) = \{U \in \mathrm{U}(N) | \det(U) = 1\} \tag{4.3.3}$$

is a Lie group. Notice that the determinant of a unitary matrix can be any complex number of norm one, not just a sign as in the orthogonal case. Consider the map:

$$\begin{array}{ccc} \mathrm{SU}(N) \times \mathrm{U}(1) & \to & \mathrm{U}(N) \\ (U, e^{i\phi}) & \mapsto & U e^{i\phi} \ . \end{array} \tag{4.3.4}$$

This map is a homomorphism. It is not a isomorphism, however, because $(\omega_N 1_N, \omega_N^{-1}) \mapsto 1_N$*, where* $\omega_N \equiv e^{2\pi i/N}$ *as usual. We can remedy to this by quotienting (as in definition 2.11) by the subgroup* $\mathbb{Z}_N = \{(\omega_N 1_N, \omega_N^{-1})\}$*. So we have an isomorphism:*

$$\mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_N \cong \mathrm{U}(N) \ . \tag{4.3.5}$$

To study Lie groups more closely, let us "zoom in" close to the identity. One way to do this is to look at the tangent space $T_e G$ to G at the identity (which is a particular point in G). Let us identify concretely this vector space for $\mathrm{SO}(N)$ and $\mathrm{U}(N)$.

Example 4.7. *Let us first consider* $\mathrm{Gl}(N, \mathbb{R})$*, and ask what is* $T_e \mathrm{Gl}(N, \mathbb{R})$*.*

We will first do this at an intuitive level, looking at elements of the group which are very close to the identity. This means considering matrices

$$M = 1 + \epsilon\, m + \ldots \tag{4.3.6}$$

where ϵ is a small number. We notice that the condition that $\det(M) \neq 0$ is automatically satisfied for small ϵ: $\det(M)$ cannot suddenly jump from 1 (for $\epsilon = 0$) to 0.

We defined the tangent space $T_p M$ using curves through p. Let us consider a curve in the space of invertible matrices, $M(t)$, that goes through the identity at $t = 0$: namely, $M(0) = 1$. The tangent space at the identity, $T_e \mathrm{Gl}(N, \mathbb{R})$, is the space of first derivatives $\dot{M}(0)$. Since the derivative is computed by taking a small t, we can apply our observation above: we deduce that $T_e \mathrm{Gl}(N, \mathbb{R})$ is just the space of all square matrices:

$$\mathrm{gl}(N, \mathbb{R}) \equiv T_e \mathrm{Gl}(N, \mathbb{R}) = \mathrm{Mat}(N, \mathbb{R}) \tag{4.3.7}$$

Similar considerations apply to $\mathrm{Gl}(N, \mathbb{C})$.

Example 4.8. *Let us now consider $\mathrm{SO}(N)$, and ask what is $T_e \mathrm{SO}(N)$. Again, we will first look at this at an intuitive level, and then formalize the computation better.*

Let us then again look at elements of the group which are very close to the identity. This means considering an orthogonal matrix $O_A = 1 + \epsilon A + \ldots$, where ϵ is a small number. What is the condition on A such that O_A is orthogonal?

$$1 = O_A O_A^t = (1 + \epsilon A + \ldots)(1 + \epsilon A^t + \ldots) = 1 + \epsilon(A + A^t) + \ldots \tag{4.3.8}$$

So at first order in ϵ, we should have

$$A + A^t = 0 \ , \tag{4.3.9}$$

*which means A is antisymmetric. This suggests that the orthogonal group "looks" like the space of antisymmetric matrices, when we zoom in around the identity. This is a vector space (because sums and multiples of antisymmetric matrices are still antisymmetric), but it is **not a group**: the product of two antisymmetric matrices is not necessarily antisymmetric:*

$$(A_1 A_2)^t = A_2^t A_1^t = (-A_2)(-A_1) = A_2 A_1 \neq -A_1 A_2 \ . \tag{4.3.10}$$

In other words:

$$A_1, A_2 \in o(N) \quad \not\Rightarrow \quad A_1 A_2 \in o(N) \ . \tag{4.3.11}$$

Let us now redo this computation in a more formal way. We defined the tangent space T_pM using curves through p. Let us consider a curve in the space of orthogonal matrices, $O(t)$, that goes through the identity at $t = 0$: namely, $O(0) = 1$. The tangent space at the identity, $T_e\mathrm{O}(N)$, is the space of first derivatives $\dot{O}(0)$. But:

$$0 = \partial_t(1) = \partial_t(O(t)O^t(t))_{t=0} = \dot{O}(0)O^t(0) + O(0)\dot{O}^t(0) = A + A^t \ . \tag{4.3.12}$$

So we obtain that $\dot{O}$ is antisymmetric. (This is of course essentially the same as the computation in (4.3.8).) So the tangent space at the identity is the space of antisymmetric matrices:

$$\mathrm{so}(N) \equiv T_e\mathrm{SO}(N) = \{A \in \mathrm{Mat}(N, \mathbb{R}) | A + A^t = 0\} \ . \tag{4.3.13}$$

Notice that the dimension of this space is easily obtained by counting the upper-diagonal elements, which are $\sum_{k=1}^{N-1} = N(N-1)/2$, in agreement with (4.3.2). This was expected: a tangent space T_pM has the same dimension as the manifold M.

Example 4.9. *For* U(N)*, a similar computation as the one in the previous example reveals that*

$$\mathrm{u}(N) \equiv T_e\mathrm{U}(N) = \{A \in \mathrm{Mat}(N, \mathbb{C}) | A + A^\dagger = 0\} \ , \tag{4.3.14}$$

the space of antihermitian *matrices. This now gives us a way to determine the dimension easily. The N diagonal entries of a hermitian matrix has real, and the $\sum_{k=1}^{N-1} = N(N-1)/2$ elements in its upper-triangular part are* complex. *The elements on its lower-triangular part are determined by those in the upper-triangular part. So the real dimension is*

$$\dim_{\mathbb{R}}(\mathrm{u}(N)) = N + 2 \times \frac{1}{2}N(N-1) = N^2 \ . \tag{4.3.15}$$

Example 4.10. *For* Sl(N) *defined in (4.3.1), at first order we have [exercise!]*

$$\det(1 + \epsilon A) = 1 + \epsilon \mathrm{Tr}(A) + \ldots \tag{4.3.16}$$

So we have

$$\mathrm{sl}(N) \equiv T_e\mathrm{Sl}(N) = \{A \in \mathrm{gl}(N) = \mathrm{Mat}(N) | \mathrm{Tr}(A) = 0\} \ . \tag{4.3.17}$$

Recall that SU(N) *is the subgroup of* U(N) *defined as the unitary matrices of* $\det = 1$*, (4.3.3). Then we also have*

$$\mathrm{su}(N) \equiv T_e\mathrm{SU}(N) = \{A \in \mathrm{u}(N) | \mathrm{Tr}(A) = 0\} \ . \tag{4.3.18}$$

Let us now look more specifically at low dimension:

Example 4.11. *The groups* SO(2) *and* U(1) *are isomorphic, under the map*

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto e^{i\theta} \ . \tag{4.3.19}$$

The tangent space to $\mathrm{SO}(2) \cong \mathrm{U}(1)$ *at the identity* e *in this case is a one-dimensional vector space* $\mathrm{so}(2) \cong \mathrm{u}(1)$*; we can think of the single generator of this vector space as either the antisymmetric matrix* $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *or as the complex number* i*.*

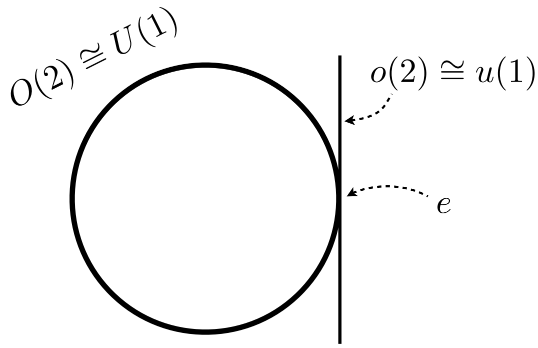


Figure 8: This figure is kind of silly: we show $\mathrm{O}(2) \cong \mathrm{U}(1)$ and its tangent space at the identity.

After definition 4.9, we mentioned (informally) that a manifold M "looks like" T_pM very close to p. We also explained that defining a map from T_pM to M would need a definition of "covariantly constant" vectors, which I left to the General Relativity class. Actually, for Lie groups there is a particularly natural way of extending an element $v_e \in T_eG$ to a vector field on G, which doesn't involve covariantly constant vectors.

Definition 4.11. *A vector field* v *on* G *is called* left-invariant *if it is invariant under the diffeomorphisms*

$$\begin{aligned} L_{g_0} : G &\to G \\ g &\mapsto g_0 g \end{aligned} \tag{4.3.20}$$

for any $g_0 \in G$*.*

To know what invariant means, we have to remember how a diffeomorphism acts on a vector: this was given on a chart by equation (4.2.7).

This definition looks too abstract. The good news is that we will rarely need it. Let's immediately look at an example:

Example 4.12. • *Let us go back to our example* $\mathrm{SO}(N)$*. Suppose we are given an element* $A \in T_e\mathrm{SO}(N)$*. We know from (4.3.13) that* A *is an antisymmetric matrix. How do we extend this to a vector field on all of* $\mathrm{SO}(N)$*? We should find some* $v(g)$ *such that* $v(e) = A$*. I will now show you what it means to extend it in a left-invariant way. Consider the choice* $v(g) = gA$*. The equation that we should solve to find the exponential is (4.2.10), which now reads*

$$\dot{g} = gA \; . \tag{4.3.21}$$

This equation is left-invariant in the sense that it remains invariant under the diffeomorphisms (4.3.20) from G *to itself, for any constant* $g_0 \in G$*. Indeed, we get* $g_0\dot{g} = g_0 gA$*, and if we multiply from the left by* g_0^{-1}*, we get (4.3.21) again.*

Let us now call $g = O(t)$*, and find the integral curve, that is, the solution to (4.3.21), which now reads* $\dot{O}(t) = O(t)A$ *with* $O(0) = 1$*. It is*

$$O(t) = \exp(tA) \equiv 1 + tA + \frac{1}{2}t^2A^2 + \ldots = \sum_{k=1}^{\infty} \frac{t^k}{k!}A^k \; ; \tag{4.3.22}$$

it solves the equation, formally for the same reasons that $\dot{x} = ax$ *is solved by* $x = e^{at}$*, with* $x \in \mathbb{R}$*. Let us check that* $O(t)$ *is orthogonal for all* t*:*

$$O(t)O^t(t) = \exp(tA)\exp(tA)^t = \exp(tA)\exp(tA^t) = \exp(tA)\exp(-tA) = \exp((t-t)A) = 1 \; . \tag{4.3.23}$$

• *We can also use the exponential parametrization* $M(t) = \exp(tm)$ *for* $\mathrm{Gl}(N)$*. Then* m *is just an arbitrary matrix, because of (4.3.7). If we use the same parametrization for* $\mathrm{Sl}(N)$*, however, (4.3.18) would lead us to expect* $\mathrm{Tr}(m) = 0$*. Indeed one can show, for any matrix* m*:*

$$\det(e^{tm}) = e^{t\mathrm{Tr}(m)} \tag{4.3.24}$$

so that $e^{tm} \in \mathrm{Sl}(N) \Leftrightarrow m \in \mathrm{sl}(N)$*.*

• *In the unitary group* $\mathrm{U}(N)$*, we have:*

$$U(t)U(t)^\dagger = 1 \; , \qquad U(t) = \exp[itH] \; , \qquad H^\dagger = H \; . \tag{4.3.25}$$

(From (4.3.14) we expect an antihermitian matrix in the exponent; indeed, iH *is antihermitian if* H *is hermitian.) You should be familiar with this: in quantum mechanics, the Hamiltonian is hermitian, and the time evolution* e^{itH} *is unitary. The same is true for other observables, of course.*

So we have in general a way of parameterizing a neighborhood of G close to the identity e by elements of T_eG. We are associating an element of the group g with a tangent vector $v_e \in T_eG$, via the exponential mapping. Since the dimension of G and of T_eG are the same, this parameterization is one-to-one, at least if we are close enough to the identity. In the case of SO(N), this parametrization is (4.3.22); in the case of U(N), by (4.3.25). can be used to parametrize the groups SO(N) and U(N).

But T_eG is not a group, it is merely a vector space. Where is the information about the product of G hidden in T_eG? Before we deal with this question in general, let us find out in a concrete example, our old friend SO(N).

4.4 Product of exponentials in the orthogonal group

In SO(N), let us compute a product of two group elements in the exponential parametrization, e^{tA_1}, $e^{tA_2} \in$ SO(N). At second order:

$$\begin{aligned} e^{tA_1}e^{tA_2} &= \left(1 + tA_1 + \frac{1}{2}t^2A_1^2 \ldots\right)\left(1 + tA_2 + \frac{1}{2}t^2A_2^2 + \ldots\right) \\ &= 1 + t(A_1 + A_2) + \frac{1}{2}t^2(A_1^2 + A_2^2 + 2A_1A_2) + \ldots \\ &= 1 + t(A_1 + A_2) + \frac{1}{2}t^2((A_1 + A_2)^2 + [A_1, A_2]) + \ldots \\ &= \exp\left(t(A_1 + A_2) + \frac{1}{2}t^2[A_1, A_2] + \ldots\right) \ , \end{aligned} \tag{4.4.1}$$

where we defined the *commutator*

$$[A_1, A_2] \equiv A_1A_2 - A_2A_1 \ . \tag{4.4.2}$$

This computation (which you have already seen in your quantum mechanics class) is the second-order expansion of the so-called *Baker–Campbell–Hausdorff* (BCH) formula. It shows that, in order to reconstruct the group structure from T_eG, we need to know the commutator of any two elements in T_eG. You might wonder whether the exponent $t(A_1 + A_2) + t^2[A_1, A_2] + \ldots$ is an element of T_eG. It better be so, because SO(N) is a group, and the product of two elements e^{tA_1} and e^{tA_2} should still be orthogonal. The linear term $t(A_1 + A_2)$ is the sum of two antisymmetric matrices: so it is still antisymmetric. What about the quadratic term, the commutator? let's see:

$$[A_1, A_2]^t = A_2^tA_1^t - A_1^tA_2^t = (-A_2)(-A_1) - (-A_1)(-A_2) = A_2A_1 - A_1A_2 = -[A_1, A_2] \ ; \tag{4.4.3}$$

in other words,
$$A_1, A_2 \in o(N) \quad \Rightarrow \quad [A_1, A_2] \in o(N) \ . \tag{4.4.4}$$
This is in contrast with (4.3.10). So it is good that the BCH formula contains a commutator and not a product. If we go at higher orders in t, we see that this is still the case. For example, at third order:
$$e^{tA_1}e^{tA_2} = \exp\left(t(A_1 + A_2) + \frac{1}{2}t^2[A_1, A_2] + \frac{1}{12}t^3([A_1, [A_1, A_2]] + [A_2, [A_2, A_1]]) + \dots\right) \tag{4.4.5}$$
The all-order expression for BCH is complicated enough that we will not give it here. In most situations, it is enough to remember the *Hadamard lemma*
$$\begin{aligned} e^{A_1}A_2e^{-A_1} &= A_2 + [A_1, A_2] + \frac{1}{2}[A_1, [A_1, A_2]] + \dots \\ &= A_2 + \mathrm{ad}_{A_1}(A_2) + \frac{1}{2}\mathrm{ad}_{A_1}(\mathrm{ad}_{A_1}(A_2)) + \dots = \exp[\mathrm{ad}_{A_1}](A_2) \end{aligned} \tag{4.4.6}$$
where we have defined the operator
$$\mathrm{ad}_{A_1}(A_2) \equiv [A_1, A_2] \ . \tag{4.4.7}$$
From the conceptual point of view, what we need to know about the BCH formula is that it can be expressed in terms of commutators only (not products, nor anticommutators) *at all orders*. Because of this, for $G = \mathrm{SO}(N)$, we can reconstruct G from T_eG, and the product in G from the commutator $[,] : T_eG \times T_eG \to T_eG$. We will now try to generalize the lessons learned in this example.

4.5 Lie algebras

In the previous subsection, we have observed that the vector space $o(N)$ of antisymmetric matrices is closed under the commutator, (4.4.4), even though it is not closed under the matrix product, (4.3.11). This motivates the following definition:

Definition 4.12. *A **Lie algebra** is a vector space $\mathfrak{g}$ over a field F*[9] *with a* bracket $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that*

- *The bracket is bilinear:* $[ax_1 + bx_2, x_3] = a[x_1, x_3] + b[x_2, x_3]$, *and* $[x_1, ax_2 + bx_3] = a[x_1, x_2] + b[x_2, x_3]$, $\forall a, b \in F$, $x_1, x_2, x_3 \in \mathfrak{g}$.
- *The bracket is antisymmetric:* $[x_1, x_2] = -[x_2, x_1]$ $\forall x_1, x_2 \in \mathfrak{g}$.

[9]We will consider $F = \mathbb{R}$ or $\mathbb{C}$, and we will speak respectively of **real** and **complex** Lie algebras.

- *The operator* $[x_1, \cdot] : \mathfrak{g} \to \mathfrak{g}$ *obeys Leibniz rule:*

$$[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_2, [x_1, x_3]] \ , \qquad \forall x_1, x_2, x_3 \in \mathfrak{g} \ . \tag{4.5.1}$$

  This is called the Jacobi identity*. It is also popular to rewrite it, using antisymmetry, as*

$$[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0 \ , \qquad \forall x_1, x_2, x_3 \in \mathfrak{g} \ . \tag{4.5.2}$$

The bracket operator is not necessarily the commutator defined in (4.4.2)! it is an abstract definition. For a general Lie algebra $\mathfrak{g}$, the products $A_1 A_2$ and $A_2 A_1$ wouldn't even be defined.

But (4.4.2) does give a class of examples:

Example 4.13. *The vector space* $\mathrm{Mat}(N, \mathbb{C})$ *is a Lie algebra, where*

$$[M_1, M_2] \equiv M_1 M_2 - M_2 M_1 \ . \tag{4.5.3}$$

More generally: given an associative algebra A *(definition 1.7) with product* $\cdot$*, it can be turned into a Lie algebra by defining* $[a_1, a_2] \equiv a_1 \cdot a_2 - a_2 \cdot a_1$*. The commutator is clearly antisymmetric and linear in both entries. What about the Jacobi identity? let's check:*

$$\begin{aligned} [[a_1, a_2], a_3] + [[a_2, a_3], a_1] + [[a_3, a_1], a_2] &= a_1 a_2 a_3 - a_2 a_1 a_3 - a_3 a_1 a_2 + a_3 a_2 a_1 \\ &\quad + a_2 a_3 a_1 - a_3 a_2 a_1 - a_1 a_2 a_3 + a_1 a_3 a_2 \\ &\quad + a_3 a_1 a_2 - a_1 a_3 a_2 - a_2 a_3 a_1 + a_2 a_1 a_3 = 0 \ . \end{aligned} \tag{4.5.4}$$

There is actually a sense in which *all* Lie algebras are of this type, as we will see in theorem 4.2.

Actually, we have seen already many more examples of Lie algebras already.

Definition 4.13. *The* ***Lie algebra*** $\mathrm{Lie}(G)$ ***of a group*** G *is the Lie algebra defined as follows. As a vector space, it is* $T_e G$*. The product is defined abstractly as follows: given* x_1*,* $x_2 \in T_e G$*, extend them to left-invariant vector fields* v_1*,* v_2 *(see definition 4.11 and example 4.12) on all of* G*. Then compute the Lie bracket* $[v_1, v_2]$ *of the two vectors, defined in definition 4.7. The value of this new vector field in* $T_e G$ *is defined to be* $[x_1, x_2]$*.*

All this looks too abstract, so let's see what it means in an example.

Example 4.14. *Again we will look at* $\mathrm{SO}(N)$*. The tangent space* $\mathrm{so}(N) \equiv T_e\mathrm{SO}(N)$ *(see (4.3.13)) is by definition the Lie algebra* $\mathrm{Lie}(\mathrm{SO}(N))$*. We now want to compute the bracket according to definition 4.13.*

We will not perform all the steps; we will take a shortcut. Instead of computing the left-invariant vector fields, we will use the exponential map found in (4.3.22), and then use the geometrical understanding of the Lie bracket discussed in the remark after definition 4.8, and illustrated in figure 6. Let us call A_1 *and* A_2 *two elements of* $o(N)$ *(they are antisymmetric matrices: see (4.3.13)). Starting from the point* $p = e$*, we want to apply the diffeomorphism* $\exp[tA_1]$ *and* $\exp[tA_2]$*, and then in the opposite order, and subtract the two results. Using (4.4.1), we see that the two differ by* $t^2[A_1, A_2] + \ldots$

We conclude that the Lie bracket in $o(N)$ *in this case is just the commutator of matrices (4.4.2). This conclusion is reassuring: we observed in (4.4.4) that* $o(N)$ *is closed under the commutator. We are left to checking the Jacobi identity. But that was checked explicitly in (4.5.4).*

Of course we skipped a few steps in the argument above, partially appealing to intuition. If you buy that logic, though, you will probably also believe that all the examples of T_eG we saw earlier work similarly:

Example 4.15. *The tangent spaces* T_eG *we saw in section 4.3 are by definition the Lie algebras Lie*(G)*:* $\mathrm{gl}(N)$*,* $\mathrm{sl}(N)$*,* $\mathrm{o}(N)$*,* $\mathrm{u}(N)$*,* $\mathrm{su}(N)$ *are all Lie algebras. The Lie bracket for all these cases is the ordinary matrix commutator.*

Remark. So from a Lie group one can define a Lie algebra. To some extent, one can also go backwards: from a Lie algebra, one can locally[10] construct a Lie group, essentially using the exponential map we have seen in example 4.12. Namely, we extend the element $x \in \mathfrak{g}$ to a left-invariant vector field, and one then applies the exponential map in definition 4.8 to obtain elements

$$e^{tx} \in G \, , \tag{4.5.5}$$

for all $t \in \mathbb{R}$. For Lie algebras expressed in terms of matrices, such as $\mathrm{gl}(N)$ and the others we have seen so far, this procedure reduces to the matrix exponential as in (4.3.22). Just as in section 4.4, we can reconstruct the product in G from the bracket in $\mathrm{Lie}(G)$ because the BCH formula (4.4.5) can be expressed in terms of commutators at all orders.

We have given an abstract definition of Lie algebras, but so far we haven't used it that much; we have only seen examples defined by matrices. Let's now introduce some examples without referring to matrices. First a useful definition:

[10] The exponential map might only reach an open subset of G; this occurs for example in $\mathrm{Sl}(2, \mathbb{R})$.

Definition 4.14. *Let $\mathfrak{g}$ be a Lie algebra, and $\{e_i\}$ a basis in it (in the vector space sense). The bracket of two elements e_i and e_j of the basis gives another element $\in \mathfrak{g}$. This can be written as a linear combination of the elements of the basis:*

$$[e_j, e_k] = f^i{}_{jk} e_i \ . \tag{4.5.6}$$

The $f^i{}_{jk}$ are called structure constants *for the Lie algebra $\mathfrak{g}$. (When we use this definition of a Lie algebra, we will usually mean that $\mathfrak{g}$ is a* real *Lie algebra, which means that we should only consider real linear combinations of the e_i.)*

One can also think of (4.5.6), and use the structure constants to *define* a bracket on a vector space, turning it into a Lie algebra. Once the bracket is defined on a basis, it can be defined by linearity on the rest of the space: given $x = x_j e_j$ and $y = y_k e_k$, we have $[x, y] \equiv e_i f^i{}_{jk} x_j y_k$. In order for the bracket to be antisymmetric, we should impose

$$f^i{}_{kj} = -f^i{}_{jk} \ . \tag{4.5.7}$$

We are left with the Jacobi identity (4.5.2), which in terms of the structure constants reads

$$f^i{}_{r[j} f^r{}_{kl]} \equiv \frac{1}{3}(f^i{}_{rj} f^r{}_{kl} + f^i{}_{rk} f^r{}_{lj} + f^i{}_{rl} f^r{}_{jk}) = 0 \ . \tag{4.5.8}$$

(Using square brackets to antisymmetrize indices is standard practice; you might find it useful in the future.) So if we have a set of $f^i{}_{jk}$ that satisfy (4.5.7) and (4.5.8) we have defined a Lie algebra.

Example 4.16.

- *Stupid example: $\mathbb{R}$ is a Lie algebra with a single generator e_1. Of course the structure constants and the Lie bracket are zero.*

- *As a first example, we can take all the $f^i{}_{jk}$ to be zero, except*

  $$f^3{}_{12} = -f^3{}_{21} = 1 \ , \tag{4.5.9}$$

  so that

  $$[e_1, e_2] = e_3 \tag{4.5.10}$$

  is the only nontrivial commutator. It is immediate to see that the Jacobi identity (4.5.2) is satisfied; indeed, $[[e_i, e_j], e_k] = 0$ for all i, j, k!

  This algebra is known as the Heisenberg algebra *for reasons that will become clear in (4.5.19).*

- *A second example with three generators ℓ_i:*

$$f^i{}_{jk} = \epsilon_{ijk} \;, \qquad [\ell_i, \ell_j] = \epsilon_{ijk}\ell_k \;. \tag{4.5.11}$$

  The Jacobi identity can be checked easily by using

$$\epsilon^{ijr}\epsilon_{rkl} = \delta^i_k\delta^j_l - \delta^i_l\delta^j_k \equiv 2\delta^i_{[k}\delta^j_{l]} \;. \tag{4.5.12}$$

  You may recognize already the algebra of the generators ℓ_i of angular momentum in three dimensions.[11]

It is natural to wonder whether these examples can be seen too in terms of matrices, even though we have not introduced them that way. Let us first formalize the question. This requires a few more definitions, very similar to the ones we introduced already for groups. For this reason, our treatment will be less thorough than the one for groups.

Definition 4.15. *A* morphism of Lie algebras *is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that*

$$\phi([x, y]) = [\phi(x), \phi(y)] \;, \qquad \forall x, y \in \mathfrak{g}_1 \;. \tag{4.5.13}$$

It is an isomorphism *if it is invertible.*

Notice the similarity with the concept of homomorphism of groups (definition 2.3). In fact, a homomorphism of Lie groups $\Phi : G_1 \to G_2$ can be used to define a morphism of the corresponding Lie algebras $\phi : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$. It might even happen that the homomorphism Φ is not an isomorphism, but ϕ is, as we will see for SU(2) and SO(3) in section 4.6.

Thanks to our abstract definition, we see that:

Lemma 4.1. *If $G_1 \cong G_2$, the Lie algebras are also isomorphic:* $\mathrm{Lie}(G_1) \cong \mathrm{Lie}(G_2)$.

The converse is not necessarily true, as is implicit in the comment around (4.5.5).

Example 4.17. *Both* U(1) *and* $\mathbb{R}$ *have Lie*$(G) \cong \mathbb{R}$.

[11] In physics, we usually prefer the operators $L_i = i\ell_i$. We will encounter similar redefinitions by factors of i later in these notes. The general reason is that the generators of our algebras will often by represented in u(N), which is the space of *antihermitian* matrices; these are the ones that close under matrix commutators. On the other hand, operators in quantum mechanics are *hermitian*. As a general rule, you can associate an operator to our generators by operator $= i$generator.

More generally, if G_1 has a discrete normal subgroup Γ,

$$G_2 \equiv G_1/\Gamma \tag{4.5.14}$$

is again a group; since Γ is discrete, G_1 and G_2 are the same near the identity, and so they share the same Lie algebra $\mathrm{Lie}(G_1) \cong \mathrm{Lie}(G_2)$. We will see an interesting example in (4.6.8).

Just like for groups, a morphism to the Lie algebra $\mathrm{gl}(N)$ deserves a special name:

Definition 4.16. *A* representation *of a Lie algebra* $\mathfrak{g}$ *is a morphism* $\rho : \mathfrak{g} \to \mathrm{gl}(N, \mathbb{C})$*. A representation is called* faithful *if the morphism* ρ *is injective.*

Notice that a representation $\rho_{\mathrm{Lie}(G)}$ of $\mathrm{Lie}(G)$ also defines a representation of the Lie group G, by simply taking the matrix exponential:

$$\rho_G(e^x) \equiv \exp(\rho_{\mathrm{Lie}(G)}(x)) \ , \tag{4.5.15}$$

where $x \in \mathfrak{g}$, and $e^x \in G$; recall that this makes sense in general, using the abstract exponential map discussed around (4.5.5). For matrix groups, e^x is just the matrix exponential.[12]

We asked earlier whether the algebras in example 4.16, which we defined abstractly by their commutation relations, could be realized in terms of matrices. A narrow sense of understanding this question is now: do there exist representations of these matrices?

Example 4.18.

- *We saw in example 4.16 that* $\mathbb{R}$ *is a Lie algebra with one generator* e_1*. Suppose we consider an infinite dimensional representation. Take the vector space of the representation to be* $V = L^2(\mathbb{R})$*, and represent*

$$\rho(e_1) = \partial_x \ . \tag{4.5.16}$$

What is the exponential of this representation? consider the Taylor expansion of a function:

$$\begin{aligned} f(x+t) &= f(x) + t(\partial_x f)(x) + \frac{1}{2}t^2(\partial_x^2 f)(x) + \dots \\ &= \left(1 + t\partial_x + \frac{1}{2}t^2 + \dots\right) f(x) = \exp[t\partial_x] f(x) \ . \end{aligned} \tag{4.5.17}$$

[12] We have seen in lemma 4.1 that two groups might have the same Lie algebra; (4.5.15) will not necessarily work for all of them. For example, it might be a representation of G_1, but not of G_2 in (4.5.14); this happens if ρ_G restricted to Γ is not the identity. We will see an example of this in example 4.18.

So the exponential of ∂_x is a finite translation, which is what we mean when we say that momentum is the generator of translations. This is a more down-to-earth way of seeing example 4.5.

- *A one-dimensional representation of $\mathfrak{g} = \mathbb{R}$ is easy to obtain: we just set $\rho(e_1) = q \in \mathbb{C}$. This can always be exponentiated as in (4.5.15) to a representation of the Lie* group $\mathbb{R}$. *However, as we noticed in example 4.17, also the Lie algebra* $\mathrm{u}(1) \cong \mathbb{R}$. *If we try to exponentiate this representation, we get $\rho_G(e^{\theta e_1}) = e^{\theta q}$. Since $e^{\theta e_1}$ is the identity for $\theta = 2\pi$, we see that $e^{2\pi q} = 1$, so that*

$$q = ki \ , \qquad k \in \mathbb{Z} \ . \tag{4.5.18}$$

So, representations of $\mathrm{U}(1)$ *are characterized by an integer k.*

Example 4.19.
- *We begin by giving an infinite-dimensional representation of the Heisenberg algebra (4.5.10), which will explain its name. Take again the vector space of the representation to be $V = L^2(\mathbb{R})$, and define*

$$\rho_{\mathrm{H}}(e_1) = \hat{x} \ , \qquad \rho_{\mathrm{H}}(e_2) = \hat{p} \ , \qquad \rho_{\mathrm{H}}(e_3) = i\hbar 1 \ , \tag{4.5.19}$$

where $\hat{x}$ is the position operator and $\hat{p}$ is the momentum operator. Now ρ_{H} is a representation because $[\hat{x}, \hat{p}] = i\hbar 1$.

- *We might prefer a finite-dimensional representation of the Heisenberg algebra. Here is an easy one:*

$$\rho(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \rho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \rho(e_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.5.20}$$

- *Let us now turn to the Lie algebra in (4.5.11). If we think of the ℓ_i as the generators of rotations, we have the following natural representation:*

$$\rho_1(\ell_1) \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad \rho_1(\ell_2) \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \qquad \rho_1(\ell_3) \equiv \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{4.5.21}$$

or in other words

$$\rho_1(\ell_i)_{jk} = -\epsilon_{ijk} \ . \tag{4.5.22}$$

One can check that the $\rho_1(\ell_i)$ satisfy the commutation relations (4.5.11). (We will explain the label $_1$ later, in case you haven't guessed it yet from your QM course.)

In fact, $\rho_1(\ell_i)$ can be seen as a basis of the space so(3) *of antisymmetric real 3×3 matrices. So we can also say that the Lie algebra* so(3) *is isomorphic to the Lie algebra defined in (4.5.11).*

- *Let us now give an infinite-dimensional representation of the algebra (4.5.11). This is given on $L^2(\mathbb{R}^3)$ by the operators*

$$\rho_v(\ell_i) = -\epsilon_{ijk} x_j \partial_k \ . \tag{4.5.23}$$

We can now check easily that these constitute a representation for the algebra (4.5.11). (Notice that these operators are actually also vector fields, and that their commutator was called their "Lie bracket" already in (4.2.9).)

This infinite-dimensional representation is actually reducible (we will find all of their representations in example 4.23, although you probably remember what they are from your QM classes). This means that there are functions which are not mixed with others by (4.5.23). For example, consider the three functions x_i:

$$\rho_v(\ell_i)(x_l) = -\epsilon_{ijk} x_j \partial_k x_l = -\epsilon_{ijk} x_j \delta_{kl} = -\epsilon_{ijl} x_j \ . \tag{4.5.24}$$

So the three functions x_i are a finite-dimensional representation (which is in fact isomorphic to ρ_1 in (4.5.21)) inside the infinite-dimensional representation ρ_v. One can actually find all the finite-dimensional irreducible representations inside ρ_v; these are nothing but the spherical harmonics*.*

We now introduce a type of representation which exists for all Lie algebras and Lie groups.

Definition 4.17. *The* ***adjoint representation*** ad *of a Lie algebra $\mathfrak{g}$ is defined by*

$$\mathrm{ad}_x(y) \equiv [x, y] \ , \qquad \forall x, y \in \mathfrak{g} \ ; \tag{4.5.25}$$

this generalizes (4.4.7). This is a representation of dimension $\dim(\mathfrak{g})$. *(It is a representation because $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$, which follows from the Jacobi identity (4.5.2)!)*

This representation can be exponentiated (in the sense of (4.5.15)) to a representation Ad *of the group G. For matrix groups, using (4.4.6), we have*

$$\mathrm{Ad}[g](y) \equiv g y g^{-1} \ , \qquad \forall g \in G, y \in \mathfrak{g} \ . \tag{4.5.26}$$

Exercise: check that (4.5.21) is the adjoint representation for the Lie algebra (4.5.11).

Definition 4.18. *A* Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ *is a subspace which is closed under the bracket of* $\mathfrak{g}$*: namely,* $[h_1, h_2] \in \mathfrak{h}$ *for all* $h_1, h_2 \in \mathfrak{h}$*. (We will also write* $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$*).*

An invariant Lie subalgebra *(or* ideal*) is a Lie subalgebra* $\mathfrak{n} \subset \mathfrak{g}$ *such that* $[\mathfrak{n}, \mathfrak{g}] = \mathfrak{n}$*: that is,* $[n, x] \in \mathfrak{n}$ *for all* $n \in \mathfrak{n}$*,* $x \in \mathfrak{g}$*.*

A Lie algebra is a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *if it is a direct sum as a vector space, and* $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$*.*

These two definitions are similar to the definitions of subgroup and of normal (or invariant) subgroup, respectively. In fact, if H is a Lie group which is a subgroup of a Lie group G, then $\mathrm{Lie}(H)$ is a subalgebra of $\mathrm{Lie}(G)$; if moreover $H \triangleleft G$, $\mathrm{Lie}(H)$ is invariant in $\mathrm{Lie}(G)$.

Let us now give a more general answer to the question of whether a Lie algebra can be represented in terms of matrices:

Theorem 4.2. ***(Ado).*** *Any finite-dimensional Lie algebra* $\mathfrak{g}$ *has a faithful representation (see definition 4.16). In other words, any* $\mathfrak{g}$ *is isomorphic to a Lie subalgebra of* $\mathrm{gl}(N)$*.*

Once again we see how an abstract definition doesn't always generalize a more concrete one — in this case, the abstract definition of a Lie algebra using generators can be reconduced to a more concrete definition in terms of matrices, at least when the generators are finitely many.

4.6 SU(2) and SO(3)

We will now look at the Lie algebra $\mathrm{su}(2) = T_e\mathrm{SU}(2)$. From (4.3.18) we see that it consists of all 2×2 complex matrices which are *anti*hermitian and traceless. Since $\mathrm{u}(2)$ has real dimension 4 (see (4.3.15)), $\mathrm{su}(2)$ has real dimension 3. A popular basis for this vector space is given by

$$\tau_i = -\frac{i}{2}\sigma_i \ , \tag{4.6.1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.6.2}$$

are called *Pauli matrices*, as you have seen in your QM course. These satisfy the relations

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij} 1_2 \ . \tag{4.6.3}$$

From this it follows that

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{4.6.4}$$

and, curiously,

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}1_2 \ . \tag{4.6.5}$$

The reason we called (4.6.5) curious is that at this point it has no reason to be true. The structure of Lie algebra only requires that the space should be closed under bracket [,], not under matrix multiplication. We will recycle this equation later (in section 4.10.2, when discussing the spinor representation). On the other hand, equation (4.6.4) is expected: it implies that su(2), spanned by the τ_i in (4.6.1), is closed under the commutator (which we knew already). Moreover, we also see that the τ_i satisfy the commutation relation (4.5.11).

So we can say that the Lie algebra su(2) is isomorphic to the Lie algebra defined in (4.5.11). This is also isomorphic to the Lie algebra of so(3) (as we noticed after (4.5.21)). In other words,

$$\mathrm{su}(2) \cong \mathrm{so}(3) \ , \tag{4.6.6}$$

because (4.6.1) and (4.5.21) both have the commutation relations (4.5.11).

A slightly different perspective is that the (4.6.1) give a representation of the Lie algebra (4.5.11):

$$\rho_{1/2}(\ell_i) = \tau_i \ . \tag{4.6.7}$$

The label $_{1/2}$ comes from the fact that in QM you have used these τ_i to represent angular momentum in spin 1/2. Because of this, we will sometimes call this the spinorial representation. (From the point of view of su(2), it might also be called the fundamental representation, just like in example 2.7.) The representation (4.5.21) will be called the vector representation.

So we can think about the isomorphism (4.6.6) as the map between the two different representations $\rho_{1/2}$ and ρ_1 of the same abstract Lie algebra $[\ell_i, \ell_j] = \epsilon_{ijk}\ell_k$. To understand the relation between SU(2) and SO(3), we can just exponentiate the two representations, as in (4.5.15), and compare the two results. In other words, we should study the map

$$e^{\rho_{1/2}(\theta_i\ell_i)} = \exp\left[\frac{1}{2}\begin{pmatrix} i\theta^3 & \theta^2 + i\theta^1 \\ -\theta^2 + i\theta^1 & -i\theta^3 \end{pmatrix}\right] \longmapsto e^{\rho_1(\theta_i\ell_i)} = \exp\begin{pmatrix} 0 & -\theta^3 & \theta^2 \\ \theta^3 & 0 & -\theta^1 \\ -\theta^2 & \theta^1 & 0 \end{pmatrix} . \tag{4.6.8}$$

Let us first compute $e^{\rho_{1/2}(\theta_i \ell_i)}$. The group SU(2) can be written as the group of matrices of the form

$$U = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix} , \qquad |Z|^2 + |W|^2 = 1 . \tag{4.6.9}$$

One can check directly that U is unitary and has determinant equal to one. Now, if we write

$$Z = X^4 + iX^3 , \qquad W = X^2 + iX^1 , \tag{4.6.10}$$

we see that SU(2) is the set $\{(X^1, X^2, X^3, X^4) \in \mathbb{R}^4 \text{such that } \sum_{i=1}^4 (X^i)^2 = 1\}$. By our definition in (4.1.1), this is a sphere S^3. It can be shown that this identification is actually a diffeomorphism. So we can write:

$$\mathrm{SU}(2) \cong S^3 \tag{4.6.11}$$

where now $\cong$ denotes the existence of a diffeomorphism.

Curiously, if we use (4.6.10), we can rewrite (4.6.9) as

$$U = \begin{pmatrix} X^4 + iX^3 & X^2 + iX^1 \\ -X^2 + iX^1 & X^4 - iX^3 \end{pmatrix} = X^4 1_2 + iX^i \sigma_i , \qquad i = 1, 2, 3 . \tag{4.6.12}$$

There is another way to check that $X^4 1_2 + iX^i \sigma_i$ is unitary. Notice first that

$$(X^i \sigma_i)^2 = X^i X^j \sigma_i \sigma_j = X^i X^j (i\epsilon_{ijk}\sigma_k + \delta_{ij} 1_2) = X^i X^i 1_2 \equiv \vec{X}^2 1_2 . \tag{4.6.13}$$

We have used (4.6.3); the product $X^i X^j \epsilon_{ijk}$ vanishes because $X^i X^j$ is symmetric under exchange $i \leftrightarrow j$, whereas ϵ_{ijk} is antisymmetric. (You will use this trick often.) We can now compute

$$\begin{aligned} U^\dagger U &= (X^4 1_2 + iX^i\sigma_i)^\dagger (X^4 1_2 + iX^i\sigma_i) = (X^4 1_2 - iX^i\sigma_i)(X^4 1_2 + iX^i\sigma_i) \\ &= (X^4)^2 + X^i X^j \sigma_i \sigma_j = (X^4)^2 + \vec{X}^2 = |Z|^2 + |W|^2 = 1 . \end{aligned} \tag{4.6.14}$$

It might be a bit confusing that we're now writing U, which is in the Lie group SU(2), as a linear combination of the Pauli matrices (or more precisely the τ_i in (4.6.1)), which are a basis for the Lie algebra su(2). The reason for this is a special circumstance for this particular Lie algebra: namely, equation (4.6.3). The antisymmetric part of that equation is (4.6.4), that guarantees that su(2) is a Lie algebra. On the other hand, there is no reason at this point that the symmetric part, namely (4.6.5), should be true. Nevertheless, let us take advantage of it and compute the exponential of an element $u \in$ su(2), parameterized, as in the left hand side of (4.6.8), as a linear combination of the τ_i in (4.6.1):

$$u = (-i/2)\theta^i \sigma_i . \tag{4.6.15}$$

Recalling (4.6.13), we get:

$$\begin{aligned} e^u = \exp[-i\theta^i\sigma_i/2] &= 1_2 + (-i\theta^i\sigma_i/2) + \frac{1}{2}(-i\theta^i\sigma_i/2)^2 + \frac{1}{6}(-i\theta^i\sigma_i/2)^3 + \dots \\ &= 1_2 + (-i\theta^i\sigma_i/2) - \frac{1}{2}\vec{\theta}^2 1_2 - \frac{1}{6}\vec{\theta}^2(-i\theta^i\sigma_i/2) + \dots \\ &= \cos(|\theta|/2)1_2 - i\frac{\theta^i}{|\theta|}\sigma_i \sin(|\theta|/2) \ , \end{aligned} \tag{4.6.16}$$

where we defined

$$|\theta| \equiv \sqrt{\vec{\theta}^2} \ . \tag{4.6.17}$$

Indeed the expression for e^u we obtained in (4.6.16) is of the form (4.6.12), where $X^4 = \cos(|\theta|/2)$, $X^i = -\frac{\theta^i}{|\theta|}\sin(|\theta|/2)$; it is easy to check that $\sum_{i=1}^4 (X^i)^2 = 1$, as it should be. The fact that the exponential e^u can be resummed explains why the Pauli matrices can be used to expand elements of the Lie group.

We should now exponentiate the representation ρ_1; namely, compute $\exp[\rho_1(\theta^i\ell_i)]$. Actually, we will not try to compute this exponential explicitly here (it is possible, but it is more complicated than in (4.6.16)). We will instead analyze the crucial features of the result using some particularly simple examples. For $\theta_1 = \theta_2 = 0$:

$$\exp[\rho_1(\theta^3 e_3)] = \exp\begin{pmatrix} 0 & -\theta^3 & 0 \\ \theta^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta^3) & -\sin(\theta^3) & 0 \\ \sin(\theta^3) & \cos(\theta^3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4.6.18}$$

In this particular case, the exponential $e^{i\theta^i\sigma_i/2} = e^{i\theta^3\sigma_3/2}$ is also very easy: it's an exponential of a diagonal matrix, which can be computed element by element — we don't even need our general formula (4.6.16). So for $\theta^1 = \theta^2 = 0$ the homomorphism (4.6.8) reads

$$\begin{pmatrix} e^{i\theta^3/2} & 0 \\ 0 & e^{-i\theta^3/2} \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta^3) & -\sin(\theta^3) & 0 \\ \sin(\theta^3) & \cos(\theta^3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4.6.19}$$

Take now $\theta^3 = 2\pi$. We have

$$-1_2 \longmapsto 1_3 \ . \tag{4.6.20}$$

So the homomorphism (4.6.8) is not a bijection. This is the famous fact that a spinor doesn't get to itself after a 2π rotation! In retrospect, it was enough to look at the general formula (4.6.16): any rotation of 2π in any direction gives a $\vec{\theta}$ such that $|\theta| = 2\pi$; then $\sin(|\theta|/2) = 0$, but $\cos(|\theta|/2) = -1$. So any rotation of angle 2π has -1_2 as a counterimage.

To summarize, we can say that the homomorphism (4.6.8) is two-to-one:

$$\mathrm{SU}(2) \xrightarrow{2:1} \mathrm{SO}(3) \; . \tag{4.6.21}$$

4.7 A cross-check

Let us see the isomorphism (4.6.6) between the Lie algebras su(2) and so(3) in yet another way. We mentioned after definition 4.17 that ρ_1 is the adjoint representation of (4.5.11), which we now know to be the algebra of su(2). We can now check this explicitly:

$$\mathrm{ad}_{-i\theta^i\sigma_i/2}(\sigma_j) = -\frac{i}{2}\theta^i[\sigma_i, \sigma_j] = \theta^i\epsilon_{ijk}\sigma_k = -\rho_1(\theta^i\ell_i)_{jk}\sigma_k \; . \tag{4.7.1}$$

So the isomorphism between su(2) and so(3) is in fact just the adjoint representation ad.

This isomorphism can be exponentiated to a homomorphism from SU(2) to SO(3):

$$\mathrm{Ad}: \; \mathrm{SU}(2) \to \mathrm{SO}(3) \; . \tag{4.7.2}$$

We would like to know whether this is an isomorphism. In general, Ad is given by (4.5.26). In slightly more concrete terms, if $g = e^{-u}$ is our element of SU(2), we can compute its Ad action on a generator $\tau_i = (-i/2)\sigma_i$ of su(2), $\mathrm{Ad}_{e^{-u}}(\tau_i) = e^{-u}\tau_i e^u$. This will give a certain linear combination of the τ_i:

$$e^{-u}\tau_i e^u = M_{ij}\tau_j \; . \tag{4.7.3}$$

Ad_{e^u} will be the matrix M_{ij}. In more physical terms, consider a spinor ψ, and suppose one wants to measure its spin expectation value:

$$s_i = \frac{1}{2}\psi^\dagger\sigma_i\psi \; . \tag{4.7.4}$$

If we perform a rotation in space, the spinor will transform by a certain element of SU(2), $\psi \to e^u\psi$. So

$$s_i \to \frac{1}{2}\psi^\dagger e^{-u}\sigma_i e^u\psi = M_{ij}\frac{1}{2}\psi^\dagger\sigma_j\psi = M_{ij}s_j \; . \tag{4.7.5}$$

So M_{ij} is just the space rotation associated to a certain rotation on the spinor. This is the physical interpretation of Ad in this case.

Let us now compute Ad. We use (4.4.6):

$$\begin{aligned} e^{i\theta^i\sigma_i/2}\sigma_j e^{-i\theta^i\sigma_i/2} &= \sigma_j + \mathrm{ad}_{i\theta^i\sigma_i/2}(\sigma_j) + \frac{1}{2}\mathrm{ad}_{i\theta^i\sigma_i/2}(\mathrm{ad}_{i\theta^i\sigma_i/2}(\sigma_j)) + \ldots = \\ &= \sigma_j + \rho_1(\theta^i\ell_i)_{jk}\sigma_k + \frac{1}{2}(\rho_1(\theta^i\ell_i))^2_{jk}\sigma_k \\ &= \exp[\rho_1(\theta^i\ell_i)]_{jk}\sigma_k \; . \end{aligned} \tag{4.7.6}$$

This gives again the map (4.6.8), now interpreted as the adjoint representation Ad for SU(2).

4.8 Complexification; unitary representations

So far, we have been looking at Lie algebras as *real* vector spaces, meaning that we allowed ourselves to take only real combinations of the generators. For example, we remarked this in our definition 4.14 of structure constants. Also, some of our Lie algebras are real vector spaces, not complex ones. For example, the Lie algebra $\mathrm{u}(N)$ was computed to be in (4.3.14) to be the vector space of antihermitian matrices. If we multiply an antihermitian matrix by a complex number, it won't be antihermitian any more. So $\mathrm{u}(N)$ is a real Lie algebra.

On the other hand, we can just decide to extend a given Lie algebra by admitting complex combinations. As a real vector space, this will be a new, larger Lie algebra. To formalize this, let us first recall another definition you have seen in QM:

Definition 4.19. *Given two vector spaces V_1 and V_2, their* tensor product $V_1 \otimes V_2 \equiv V_1 \times V_2/\sim$ *is their cartesian product quotiented by an equivalence relation*

$$(\lambda v_1, v_2) \sim (v_1, \lambda v_2) \ , \qquad \lambda \in \mathbb{C} \ , \ v_i \in V_i \ . \tag{4.8.1}$$

The equivalence class of (v_1, v_2) is denoted by $v_1 \otimes v_2$.

$V_1 \otimes V_2$ is a vector space of dimension $\dim(V_1) \times \dim(V_2)$. *Given a basis $\{v_{1i}\}$ for V_1 and a basis $\{v_{2i}\}$ for V_2, the vectors $v_{1i} \otimes v_{2j}$ provide a basis for $V_1 \otimes V_2$.*

This is doing nothing but formalizing a definition you've always used: practically, you knew already $(\lambda v_1) \otimes v_2 = v_1 \otimes (\lambda v_2) \equiv \lambda v_1 \otimes v_2$. Notice that we didn't specify whether V_i were real or complex vector spaces, but we took $\lambda \in \mathbb{C}$. So:

Definition 4.20. *Given a real vector space V, its* complexification *is defined to be $V_{\mathbb{C}} \equiv V \otimes \mathbb{C}$. In other words, if V is the real span of a basis e_i, $V_{\mathbb{C}}$ is the complex span of the same basis e_i. We also say that V is a* real form *of $V_{\mathbb{C}}$.*

This definition is just a way of allowing you to multiply real vectors by complex numbers: λv is understood as $v \otimes \lambda$. Using the concept of tensor product is a bit of an overkill here, but we are going to use it later.

Considering the complexification of a Lie algebra is sometimes very useful; for example, if we want to look for representations more systematically. You actually have done so already, when looking at representations of angular momentum:

Example 4.20. *Let us consider again the Lie algebra (4.5.11). Its complexification* $\mathrm{su}(2)_{\mathbb{C}}$ *is just the complex vector space spanned by the ℓ_i: we are now allowed to take complex* *combinations of the* ℓ_i*. In particular, we can now form the combinations*

$$X \equiv i\ell_+ \equiv \frac{i}{\sqrt{2}}(\ell_1 + i\ell_2) \ , \qquad Y \equiv i\ell_- \equiv \frac{i}{\sqrt{2}}(\ell_1 - i\ell_2) \ , \qquad H \equiv i\ell_3 \ . \tag{4.8.2}$$

These satisfy the commutation relations

$$[H, X] = X \ , \qquad [H, Y] = -Y \ , \qquad [X, Y] = H \ . \tag{4.8.3}$$

This algebra is isomorphic to the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ *of (complex)* 2×2 *traceless matrices:*

$$X \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad Y \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \qquad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{4.8.4}$$

So we can say

$$\mathrm{su}(2)_{\mathbb{C}} \cong \mathrm{sl}(2,\mathbb{C}) \ . \tag{4.8.5}$$

In fact, we can also consider the matrices in (4.8.4) as a basis for the space of real 2×2 *traceless matrices. So we also have:*

$$\mathrm{sl}(2,\mathbb{R})_{\mathbb{C}} \cong \mathrm{sl}(2,\mathbb{C}) \ . \tag{4.8.6}$$

This is somewhat obvious: the complexification of the space of real traceless matrices is the space of complex traceless matrices. So both $\mathrm{sl}(2,\mathbb{R})$ *and* $\mathrm{su}(2)$ *are real forms of* $\mathrm{sl}(2,\mathbb{C})$*.*

One last ***remark****. In this example, we have seen* $\mathrm{sl}(2,\mathbb{C})$ *as a complex Lie algebra with three generators, but we can also view it as a* real *Lie algebra with six generators. This is done by simply promoting* ie_i *to new generators* f_i*. It might seem a play with words, but the distinction is important when considering representations. If one considers* $\mathrm{sl}(2,\mathbb{C})$ *as a complex Lie algebra, a representation should be complex-linear, so that*

$$\rho(ie_i) = i\rho(e_i) \ . \tag{4.8.7}$$

If, on the other hand, one considers $\mathrm{sl}(2,\mathbb{C})$ *as a real Lie algebra,* $f_i = ie_i$ *is just another generator, unrelated to* e_i*, so that in general*

$$\rho(f_i) \neq i\rho(e_i) \ . \tag{4.8.8}$$

This comment will become important (and perhaps clearer) in section 4.10.

The combinations X, Y in (4.8.2) are useful because they are "creation" and "annihilation" operators, that allow us to construct representations. You have seen this already in QM, but for completeness we repeat it here quickly.

Example 4.21. *We will now review the finite-dimensional irreducible representations of* $\mathrm{sl}(2,\mathbb{C})$ *(viewed as a complex Lie algebra; see comment at the end of example 4.20).*

(Throughout this example, we will write H rather than $\rho(H)$, and so on. The use of Dirac's notation for the states should make it clear that we're dealing with a representation.)

Notice first of all that, if $|m_0\rangle$ is an eigenvector of H,

$$H|m_0\rangle = m_0|m_0\rangle \ , \tag{4.8.9}$$

because of (4.8.3), $X|m_0\rangle$ is an eigenvector with eigenvalue m_0+1:

$$H(X|m_0\rangle) = (X + Xm_0)|m_0\rangle = (m_0+1)X|m_0\rangle \ . \tag{4.8.10}$$

We can keep generating vectors $X^k|m_0\rangle$; if the representation is to be finite-dimensional, this sequence has to stop. There has to exist an eigenvector $|l\rangle$ of H (of eigenvalue l) which is annihilated by X:

$$X|l\rangle = 0 \ , \qquad H|l\rangle = l|l\rangle \ . \tag{4.8.11}$$

Similarly, $Y|m_0\rangle$ is an eigenvector with eigenvalue m_0-1. Again, we can keep generating vectors $Y^k|h\rangle$, but if the representation is to be finite-dimensional, this sequence has to stop. There has to exist an eigenvector $|l_-\rangle$ of H (of eigenvalue l_-) which is annihilated by Y. So we have a set of eigenvectors $|m_i\rangle$ of H, with eigenvalues $l_- \le m_i \le l$. The span of these vectors is a representation: we won't generate any more vectors by acting with Y on $X^k|m_0\rangle$, for example, because $Y(X^k|m_0\rangle)$ can be reexpressed as a combination of the $X^i|m_0\rangle$ by using $[X,Y]=H$ from (4.8.3). So we can consider now the trace of H over the span of the $|m_i\rangle$. This is easy, because all of them are eigenvalues:

$$\mathrm{Tr}H = l_- + (l_-+1) + \ldots + (l-1) + l \ . \tag{4.8.12}$$

But $\mathrm{Tr}H = \mathrm{Tr}[X,Y] = 0$. So the expression we just computed should be zero. This implies that $l_- = -l$. Moreover, for states $|m_1\rangle$ and $|m_2\rangle$ in the same representation, $m_1 - m_2$ should be an integer, since they are related by powers of X. In particular, $l-(-l)$ should be an integer, which means that l is "half-integer", namely

$$l \in \frac{1}{2}\mathbb{N} \ . \tag{4.8.13}$$

(Notice that we include ordinary integers in our definition of "half-integers".)

Summing up, an irreducible, finite-dimensional representation of $\mathrm{sl}(2,\mathbb{C})$ *is spanned by eigenvectors of H:*

$$|m\rangle \ , \qquad m = -l, \ldots, l \ . \tag{4.8.14}$$

where l is half-integer; we will sometimes call it "spin", for reasons that will become clear in example 4.23. We will call m the "weight", and $|\pm l\rangle$ the highest-weight (lowest-weight) state, or vector, in the representation.

An important class of representation is:

Definition 4.21. *Given a* real *Lie algebra $\mathfrak{g}$, a representation ρ is called* hermitian *if $i\rho(x)$ is a hermitian matrix $\forall x \in \mathfrak{g}$.*

Notice that it wouldn't make sense to define this for a *complex* Lie algebra: if $i\rho(x)$ is hermitian, $i\rho(ix) = -\rho(x)$ is antihermitian.

Definition 4.21 might look perverse: why do we ask for ρ to be antihermitian rather than hermitian? recall that antihermitian matrices close under the commutator, not hermitian matrices. On the other hand, we like hermitian matrices (for example, in QM) because they have real eigenvalues. This is the source of the bothersome i in many formulas in these lectures. For example, the angular momentum operators are related to the generators of the Lie algebra su(2) by $L_i = i\ell_i$.

Example 4.22. *Both representations ρ_1 in (4.5.21) and $\rho_{1/2}$ in (4.6.7), (4.6.1) are hermitian.*

Recall that a representation of Lie(G) induces a representation of G, (4.5.15). A hermitian representations induces a unitary representation of G. A unitary representation was defined in definition 3.1, but obviously the definition makes sense in general. Some of the techniques we introduced in the study of finite groups cannot be used for general Lie groups. For example, the idea of summing over the elements of the group, used in theorem 3.2, and the concept of "space of functions" $\mathbb{C}[G]$ introduced in 3.2, need to be generalized with care. Instead of summing over G, we need to be able to integrate over it. For a compact G, it turns out that there is a suitable concept of integration, called **Haar measure**. With its help, we can define the space $L^2(G)$ of square-integrable functions on G, and one can prove an analogue to theorem 3.2 and lemma 3.3:

Theorem 4.3. *Every finite-dimensional representation ρ of a* compact *Lie group G is equivalent to a unitary representation. (If ρ is also decomposable, it is reducible.)*

A unitary representation of a Lie group G gives rise to a hermitian representation of its Lie algebra Lie(G). So, if G is compact, we also have that every finite-dimensional representation of *Lie*(G) is equivalent to a hermitian representation.

Example 4.23. SU(2) *is compact because it is topologically the same as* S^3*, as we saw in (4.6.11).* SO(3) *is also compact, because of the map (4.6.8), which topologically implies* $\mathrm{SO}(3) \cong S^3/\mathbb{Z}_2$*. So we can apply theorem 4.3 to the Lie group* SU(2)*; it follows that any finite-dimensional representation of the real Lie algebra* $\mathrm{su}(2) \cong \mathrm{so}(3) \cong$ *(4.5.11) is equivalent to a hermitian representation.*

We do have finite-dimensional representations of this algebra, because we have classified in example 4.21 all finite-dimensional complex representations of its complexified version $\mathrm{su}(2)_{\mathbb{C}} \cong \mathrm{sl}(2, \mathbb{C})$*. Recall that complexification just means allowing complex combinations of the generators; if one goes back to real combinations of the generators, one gets the algebra one started with.*

Let us now look for these hermitian representations. We have to find a basis such that the representations in example (4.21) are such that $\rho(\ell_i)$ *are antihermitian:*

$$\rho(\ell_i)^\dagger = -\rho(\ell_i) \ . \tag{4.8.15}$$

(Recall that the angular momentum generators in QM are $L_i = i\ell_i$*; see remark after definition 4.21).*

From the definition (4.8.2) of X*,* Y *and* H*, we see that this implies*

$$\rho(X)^\dagger = \rho(Y) \ , \qquad \rho(H)^\dagger = \rho(H) \ . \tag{4.8.16}$$

(Just like in example 4.21, we will ***omit the*** ρ*, to make our formulas as readable as possible.)*

Here is a way to find the right basis. First we start with the state $|l\rangle$*, the state which is annihilated by* X*. Let's declare it to have norm* $|||l\rangle|| = 1$*. Let's also define*

$$|l\rangle^\dagger = \langle l| \ . \tag{4.8.17}$$

Now we can compute the norm of $Y|\ell\rangle$*. First of all, we have*

$$XY|l\rangle = (YX + [X,Y])|l\rangle = [X,Y]|l\rangle = H|l\rangle = l|l\rangle \ . \tag{4.8.18}$$

From this it follows that

$$||Y|l\rangle||^2 = \langle l|Y^\dagger Y|l\rangle = \langle l|XY|l\rangle = l\langle l|l\rangle = l \ . \tag{4.8.19}$$

We can proceed in a similar way to compute the norm of $Y^k|l\rangle$ *for all* k*.*

However, let us now ***specialize to*** $l = 1$*, to simplify our computations.*

$$XY^2|1\rangle = (YX + H)Y|1\rangle = Y(YX + H)|1\rangle = Y|1\rangle \ . \tag{4.8.20}$$

Here we have used $H(Y|1\rangle)=0$, which follows from the fact that Y is a lowering operator for H, like X is a raising operator; see (4.8.10). From (4.8.20) we also have

$$||Y^2|1\rangle||^2 = \langle 1|X^2Y^2|1\rangle = \langle 1|XY|1\rangle = ||Y|1\rangle||^2 = 1 \ . \tag{4.8.21}$$

So we can take as a basis

$$|1\rangle \ , \qquad |0\rangle \equiv Y|1\rangle \ , \qquad |-1\rangle \equiv Y^2|1\rangle \ , \tag{4.8.22}$$

which are all of norm 1.

Now we know the action of X on the basis (4.8.22), thanks to (4.8.18) and (4.8.20). We also know the action of Y, by definition, and of H again by the fact that Y is a lowering operator. Summing up:

$$\rho(X) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad \rho(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \qquad \rho(H) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ . \tag{4.8.23}$$

So we see that (4.8.16) is satisfied.

The representation (4.8.23) is actually ρ_1 in (4.5.21) in another basis. If we had done our computations for $l > 1$, we would have obtained new representations. So we have obtained many representations of the generators of angular momentum (or spin) ℓ_i. Each of these representations is labeled by a half-integer, l; from (4.8.14), we see that the possible eigenvalues m of $\rho(\ell_3)$ are then given by all m such that $-l \leq m \leq l$.

This gives an example of a possible strategy for constructing unitary representations of a compact group. We would like to know whether we can find *all* unitary representations this way. Since we have already classified all finite-dimensional representations of $\mathrm{sl}(2,\mathbb{C})$, all we need to know is whether there are any *infinite*-dimensional representation we would like to prove some kind of converse to theorem 4.3. We will again proceed in a similar way to our treatment of finite groups. We mentioned earlier that the space of functions $\mathbb{C}[G]$ that was so important for finite groups can be generalized to the infinite-dimensional space $L^2(G)$ of square-integrable functions over G. There is then an analogue of theorem 3.6 (which said that the ρ_{aij} are a basis in $\mathbb{C}[G]$):

Theorem 4.4. *(**Peter–Weyl**; see for example [2, 4]). Let G be a compact Lie group. The matrix coefficients ρ_{aij} of all irreducible unitary representations ρ_a are a complete orthogonal system in $L^2(G)$.*

We could write the orthogonality relation among the ρ_a explicitly, obtaining a result similar to (3.2.8). We will not do this here.

Using theorem 4.4, one can show:

Theorem 4.5. *A unitary irreducible representation of a compact Lie group is finite-dimensional.*

Example 4.24. *Since* $\mathrm{SU}(2)$ *is a compact group, and since we have classified all finite-dimensional representations of* $\mathrm{sl}(2,\mathbb{C})$*, the representations in example 4.23 are all the hermitian representations of* $\mathrm{su}(2)$*.*

Apart from this example, theorems 4.3 and 4.5 are not very useful if we don't have a nice way of recognizing which groups are compact. We will now give an answer to this question. First we need an important definition:

Definition 4.22. *The* ***Killing form*** *of a real Lie algebra is defined as the quadratic form*

$$k(x,y) \equiv \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y) \ . \tag{4.8.24}$$

More concretely, in terms of the structure constants $f^i{}_{jk}$ *(defined in (4.5.6)):*

$$k_{ij} \equiv k(e_i, e_j) = \mathrm{Tr}(\mathrm{ad}_{e_i} \mathrm{ad}_{e_j}) = f^l{}_{ik} f^k{}_{jl} \ . \tag{4.8.25}$$

For future reference, we also need the following nice property of the Killing form:

$$k([x,y],z) = k(x,[y,z]) \ , \qquad \forall x,\ y,\ z \in \mathfrak{g} \ . \tag{4.8.26}$$

This can be proven easily (**exercise**) using that $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$ (see definition 4.17). In terms of the structure constants, we can see that this means $f^l{}_{ij} k_{lk} = k_{il} f^l{}_{jk}$. In other words,

$$f_{kij} = f_{ijk} \ , \qquad f_{ijk} \equiv k_{il} f^l{}_{jk} \ . \tag{4.8.27}$$

Since f_{ijk} is already antisymmetric in the last two indices, we conclude that f_{ijk} is completely antisymmetric. However, working with the f_{ijk} is not advisable in general, unless one has a way of reconstructing the $f^i{}_{jk}$ from them. We will see just such a case in section 5.1.

Definition 4.23. *A Lie algebra* $\mathfrak{g}$ *is called* compact *if it has a negative-*semi*definite Killing form* k_{ij}*, and* non-compact *otherwise.*

We then have a theorem:

Theorem 4.6. *If G is a compact Lie group, its Lie algebra* $\mathrm{Lie}(G)$ *is compact.*

Example 4.25. *Consider the Lie algebra* $\mathrm{u}(1)$*. It has a single generator e_1; the structure constants are zero. Hence the Killing form is $k_{11} = 0$. This is negative semidefinite (the "semi" exactly means that zero eigenvalues are allowed!) Indeed* $\mathrm{U}(1)$ *is a compact group. However, the computation would work in the same way for the Lie group $\mathbb{R}$. This shows that the converse of theorem 4.6 does not hold. (There is a sense in which, however, $\mathbb{R}^n$ is the only counterexample.)*

Example 4.26.
- *Let us consider again the algebra* $\mathrm{su}(2)$*. The structure constants were seen in (4.5.11): we have $f^i{}_{jk} = \epsilon_{ijk}$. Using (4.5.12), we see that*

$$k_{ij} = \epsilon^l{}_{ik}\epsilon^k{}_{jl} = -2\delta_{ij} \qquad (\mathrm{su}(2)) \ . \tag{4.8.28}$$

  This agrees with the fact that $\mathrm{SU}(2) \cong S^3$ *(recall (4.6.11)) is compact.*

- *Let us now consider* $\mathrm{sl}(2,\mathbb{R})$*. This is the vector space of real traceless 2×2 matrices. We can take as generators the three matrices defined in (4.8.2). If we call them $X = e_1$, $Y = e_2$, $H = e_3$, the non-zero structure constants are*

$$1 = f^1{}_{31} = -f^1{}_{13} = -f^2{}_{32} = f^2{}_{23} = f^3{}_{12} = -f^3{}_{21} \ . \tag{4.8.29}$$

  We then compute

$$k = 2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (\mathrm{Sl}(2,\mathbb{R})) \ . \tag{4.8.30}$$

  Its eigenvalues are $(1,1,-1)$*. So it is not negative definite. Indeed* $\mathrm{Sl}(2,\mathbb{R})$ *is non-compact.*

Notice that $\mathrm{su}(2)$ *and* $\mathrm{sl}(2,\mathbb{R})$ *are both real forms of* $\mathrm{sl}(2,\mathbb{C})$*, but one is compact, the other isn't.*

As a counterpoint to theorem 4.5, let us also remark:

Theorem 4.7. *A unitary representation of a noncompact group is infinite-dimensional, with the exception of the trivial representation.*

For example, the unitary representations of $\mathrm{sl}(2,\mathbb{C})$ are known, and they are infinite-dimensional. One can for example represent it on on $L^2(\mathbb{C})$, the infinite-dimensional space of square-integrable functions on $\mathbb{C}$. The details are interesting, and involve the so-called Möbius transformations of the complex plane plus the point at infinity, but we will not give them here. For details, see for example [4, II.4].

4.9 Tensors as representations

This section will give many examples of representations, mostly for $\mathrm{SO}(N)$ and $\mathrm{SU}(N)$.

We know already a representation for $\mathrm{SO}(N)$: it is implicit in its definition. A representation is a homomorphism of the group into a group of matrices; but our definition for $\mathrm{SO}(N)$ was already in terms of matrices, acting on $\mathbb{R}^N$. In example 2.7, we called this the fundamental representation.

We have introduced in definition 4.19 the concept of tensor product of two spaces.

Example 4.27. $\mathrm{Mat}(N,\mathbb{R})$ *can be seen as the tensor product* $\mathbb{R}^N\otimes\mathbb{R}^N$*. An element* $v\otimes w$*, for example, can be seen as a matrix whose entries* $M_{ij}=v_iw_j$*. Such a matrix has rank one (**Exercise!**) Of course, a general matrix is not of this form: a tensor product space is* spanned *by elements of the form* $v\otimes w$*, but this doesn't mean that all of its elements are of this form.*

If V_1 and V_2 are two representations, $V_1\otimes V_2$ is also a representation:

Definition 4.24. *Given representations* ρ_1 *on* V_1 *and* ρ_2 *on* V_2*, the **tensor representation*** $\rho_1\otimes\rho_2$ *on* $V_1\otimes V_2$ *is given by*

$$(\rho_1\otimes\rho_2)(g)(v_1\otimes v_2)=(\rho_1(g)v_1)\otimes(\rho_2(g)v_2)\ . \tag{4.9.1}$$

Example 4.28. *Let us go back to example 4.27. On vectors, we know that a matrix* O *acts as* $v^m\to O^m{}_nv^n$*. On any vectors of the form* $v\otimes w$*, the action then is* $v^mw^n\to O^m{}_pv^pO^n{}_qw^q$*. This action will then be the same for any matrix:*

$$M^{mn}\mapsto O^m{}_pM^{pq}O^n{}_q\ . \tag{4.9.2}$$

In other words, we have a representation on $\mathbb{R}^N\otimes\mathbb{R}^N=\mathrm{Mat}(N,\mathbb{R})$*:*

$$\rho_{V\otimes V}(M)=OMO^t\ . \tag{4.9.3}$$

Is this representation irreducible? It is not, because we can find a particular matrix which is invariant:

$$\rho_{V\otimes V}(1_N)=O1_NO^t=1_N\ . \tag{4.9.4}$$

In fact, we see that the trace of a matrix is conserved:

$$\mathrm{Tr}(M)\mapsto\mathrm{Tr}(OMO^t)=\mathrm{Tr}(O^tOM)=\mathrm{Tr}(M)\ . \tag{4.9.5}$$

So the representation $V \otimes V$ splits into the space of traceless matrices Mat_0, and matrices proportional to the identity 1_N. These two spaces are not mixed by the action of $\rho_{V\otimes V}$ of any element in $\mathrm{O}(N)$.

In fact, the space of traceless matrices Mat_0 splits even further. For example, notice that an antisymmetric matrix $\mathrm{so}(N)$ is left antisymmetric. If $A = -A^t$, $\rho_{V\otimes V}(A) = OAO^t$. This matrix is still antisymmetric:

$$(OAO^t)^t = OA^tO^t = -OAO^t \ . \tag{4.9.6}$$

Similarly, symmetric matrices are left symmetric. So the representation $V \otimes V$ splits in three representations: symmetric traceless matrices Sym_0, antisymmetric matrices $\mathrm{so}(N)$, and matrices proportional to the identity. Symbolically:

$$V \otimes V = \mathrm{Sym}_0 \oplus \mathrm{so}(N) \oplus \langle 1_N \rangle \ . \tag{4.9.7}$$

Notice that the space of antisymmetric matrices $\mathrm{so}(N)$ is the adjoint representation for $\mathrm{SO}(N)$: it is the representation where the group acts directly on the Lie algebra itself, see definition 4.17.

In what follows, we will want to avoid giving new names (such as "Sym_0") to spaces of matrices all the time. A convenient notation is to denote a representation by its dimension. For example, the space Sym_0 of symmetric traceless matrices has dimension $\frac{1}{2}N(N+1)-1$, whereas the space of antisymmetric matrices $\mathrm{so}(N)$ has the dimension $\frac{1}{2}N(N-1)$. So (4.9.7) can be written as

$$N \otimes N = \left(\frac{1}{2}N(N+1)-1\right) \oplus \left(\frac{1}{2}N(N-1)\right) \oplus 1 \ . \tag{4.9.8}$$

Example 4.29. *If we specialize example 4.28 to the case $N = 3$, we see that (4.9.8) now gives*

$$3 \otimes 3 = 5 \oplus 3 \oplus 1 \ . \tag{4.9.9}$$

For $\mathrm{SO}(3)$, however, we can also label representations using spin (the half-integer l defined in (4.8.13)):

$$(s=1) \otimes (s=1) = (s=2) \oplus (s=1) \oplus (s=0) \ . \tag{4.9.10}$$

This is nothing but the rule we learn in quantum mechanics for adding angular momenta. In general, the way we add angular momenta is nothing but decomposing tensor products of representations of $\mathrm{SO}(3)$ into irreducible representations.

One can also consider tensors with many indices:

$$M^{m_1 \ldots m_k} \ . \tag{4.9.11}$$

This representation will in general not be irreducible; for example, $\delta_{m_1 m_2} M^{m_1 m_2 m_3 \ldots m_k}$ will be a sub-representation, or the totally antisymmetric part $M^{[m_1 \ldots m_k]}$, and so on.

Let us now move on to SU(N). This group also has a N-dimensional fundamental representation, on the vector space $V = \mathbb{C}^N$:

$$v^i \mapsto U^i{}_j v^j \ , \qquad U \in \mathrm{SU}(N) \ . \tag{4.9.12}$$

There is also an *antifundamental* representation $\bar{V}$, which acts on a vector by the conjugate $\bar{U}$:

$$v^{\bar{i}} \mapsto \bar{U}^{\bar{i}}{}_{\bar{j}} v^{\bar{j}} \ . \tag{4.9.13}$$

In other words, $\rho_{\bar{V}}(U) = \bar{U}$, which is trivially a representation because $\overline{(U_1 U_2)} = \bar{U}_1 \bar{U}_2$.

We have a hermitian product in $\mathbb{C}^N$:

$$v^\dagger w = \sum_{i=1}^N \overline{v^i} w^i \equiv v^{\bar{i}} \delta_{\bar{i}j} w^j \ ; \tag{4.9.14}$$

$\delta_{i\bar{j}}$ is just the Kronecker delta. A common convention is to use this symbol to lower indices:

$$v_i \equiv \delta_{i\bar{j}} v^{\bar{j}} \ , \tag{4.9.15}$$

so that

$$v^\dagger w = v_i w^i \ . \tag{4.9.16}$$

Example 4.30. • *The tensor product $V \otimes \bar{V}$ consists of tensors of the type $M^{\bar{i}j}$. These transform as*

$$M^{\bar{i}j} \mapsto \bar{U}^{\bar{i}}{}_{\bar{k}} U^j{}_l M^{\bar{k}l} \ , \tag{4.9.17}$$

or also

$$M \mapsto U M U^\dagger \ . \tag{4.9.18}$$

This representation is not irreducible, because the identity is invariant: $1_N \mapsto U 1_N U^\dagger = 1_N$. In terms of indices, this invariant tensor can be written as the $\delta_{i\bar{j}}$ appearing in (4.9.14).

On the other hand, we can no longer antisymmetrize or symmetrize in the two indices, because they are of different type: they transform differently. It doesn't make any sense to compare $M_{\bar{i}j}$ to $M_{i\bar{j}}$; they are elements of two different representations.

So in this case we can just decompose $V \otimes \bar{V}$ in the span of the identity, and traceless matrices. In the "dimension" notation we used for (4.9.8):

$$N \otimes \bar{N} = (N^2 - 1) \oplus 1 \ . \tag{4.9.19}$$

Notice that $N^2 - 1$ is the adjoint (compare with the dimension of $\mathrm{su}(N)$ *in (4.3.15)).*

- *We can also consider the tensor product $V \otimes V$. These are tensors with two indices of the same type, such as M^{ij}. These transform like:*

$$M^{ij} \to U^i{}_k U^j{}_l M^{kl} \ ; \tag{4.9.20}$$

in other words, $M \to UMU^t$. This time, the identity is not invariant; there is no analogue of $\delta_{i\bar{j}}$ where both indices are unbarred.

However, it now makes sense to divide into symmetric and antisymmetric part. So this time we have

$$N \otimes N = \left(\frac{1}{2}N(N+1)\right) \oplus \left(\frac{1}{2}N(N-1)\right) \ . \tag{4.9.21}$$

A general tensor would have many indices of both types: $M^{i_1\ldots i_m \bar{j}_1\ldots\bar{j}_n}$ (or, in the alternative notation (4.9.15), $M^{i_1\ldots i_m}_{j_1\ldots j_n}$). This transforms as

$$M^{i_1\ldots i_m \bar{j}_1\ldots\bar{j}_n} \to U^{i_1}{}_{k_1} \ldots U^{i_m}{}_{k_m} \bar{U}^{\bar{j}_1}{}_{\bar{l}_1} \ldots \bar{U}^{\bar{j}_n}{}_{\bar{l}_n} M^{k_1\ldots k_m \bar{l}_1\ldots\bar{l}_n} \ . \tag{4.9.22}$$

What a mess! we then would need to consider all possible symmetrizations, antisymmetrizations, traces. We clearly need some smarter way to decompose tensor products into irreducible representations.

Fortunately, there is a graphical trick that keeps track of all these subtleties for us. Basically, we are trying to keep track of the properties under permutations of an ordered set (the indices of a tensor). This problem is related to finding the irreducible representations of the permutation group S_N. As we sketched in section 3.5, this involves the use of Young diagrams (see definition 3.8).

First, let us consider a tensor $M^{i_1\ldots i_m}$ which is completely symmetric: namely, it is symmetric under exchange of any two indices. We will denote graphically each index by a box, so that

$$\square \qquad \longleftrightarrow \qquad \text{fundamental representation } V \ . \tag{4.9.23}$$

A completely symmetric tensor $S^{i^1\ldots i_m}$ will be denoted by a horizontal row of m boxes:

$$\underbrace{\square\square\square\square\ldots\square}_{m \text{ boxes}} \qquad \longleftrightarrow \qquad \left\{ \begin{array}{c} \text{symmetric tensors} \\ \text{with } m \text{ indices} \end{array} \right\} \ . \tag{4.9.24}$$

Similarly, a completely antisymmetric tensor $A^{i_1 \cdots i_m}$ will be denoted with a vertical column of m boxes:

$$\left.\begin{array}{|c|}\hline \\ \hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline \end{array}\right\} m \text{ boxes} \qquad \longleftrightarrow \qquad \left\{ \begin{array}{c} \text{antisymmetric tensors} \\ \text{with } m \text{ indices} \end{array} \right\} . \tag{4.9.25}$$

Notice that there is an upper bound to how many boxes we can stack vertically. There are no antisymmetric tensors of SU(N) with m indices when $m > N$, and there is only one when $m = N$. This is the ϵ tensor. It is invariant:

$$\epsilon_{i_1 \ldots i_N} \mapsto U^{i_1}{}_{j_1} \ldots U^{i_N}{}_{j_N} \epsilon_{j_1 \ldots j_N} = \det(U) \epsilon_{i_1 \ldots i_N} = \epsilon_{i_1 \ldots i_N} \ . \tag{4.9.26}$$

So the Young diagram with m vertical boxes is just the trivial representation (which we sometimes call "the singlet"). For SU(N), we are only going to use Young diagrams whose columns are at most $N - 1$ boxes deep.

A more complicated Young diagram labels a representation of tensors which are neither fully symmetric nor fully antisymmetric. Consider for example,

$$\yng(2,1) \ . \tag{4.9.27}$$

This is associated to tensors with three indices, with mixed symmetry properties under exchange of indices. Imagine to fill the boxes with indices, as in $\young(ij,k)$. Now, first symmetrize in i, j, the indices in the same row, and then antisymmetrize in i, k, the indices in the same column. In other words, consider tensors P^{ijk} of the form

$$P^{ijk} = P_{\tiny\yng(2,1)}(M)^{ijk} \equiv M^{ijk} + M^{jik} - M^{kji} - M^{jki} \ . \tag{4.9.28}$$

This is another irreducible representation. Notice the close similarity with a representation of the symmetric group S_n: see equations (3.5.4) and (3.5.5). If you find the characterization too indirect, one can also characterize it by what it isn't: it consists of tensors P^{ijk} which are antisymmetric under exchange of i and k, but which aren't in the totally antisymmetric representation, or in other words, whose totally antisymmetrization vanishes:

$$P^{ijk} + P^{jki} + P^{kij} = 0 \ . \tag{4.9.29}$$

So far we have not dealt with tensors that also have "barred" indices, such as $v^{\bar{i}}$. In the "lowered index" notation in (4.9.15), this would read v_i. We can use the ϵ tensor to reduce this case to a tensor with all indices up:

$$v^{j_1 \cdots j_{N-1}} \equiv \epsilon^{j_i \cdots j_{N-1} i} v_i = \epsilon^{j_i \cdots j_{N-1} i} \delta_{i\bar{l}} v^{\bar{l}} \ . \tag{4.9.30}$$

So the antifundamental representation $\bar{V}$ is equivalent to the $(N-1)$-th antisymmetric product of the fundamental representation, $\Lambda^{N-1}V$. Hence

$$\left.\begin{array}{|c|}\hline \\ \hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline \end{array}\right\} N-1 \text{ boxes} \qquad \longleftrightarrow \qquad \text{antifundamental representation } \bar{V} \; . \tag{4.9.31}$$

Similar manipulations can be used for tensors with many indices of both barred and unbarred type.

Example 4.31. *Let us now consider* SU(2)*. In this case, we have the peculiarity that the representation $\bar{V}$ is actually equivalent to V. This is because*

$$\bar{U} = \epsilon U \epsilon^{-1} \; , \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; , \qquad \forall U \in \mathrm{SU}(2) \; , \tag{4.9.32}$$

as can be easily checked using (4.6.9). So we can limit ourselves to considering indices i, ignoring indices $\bar{i}$. An alternative way of understanding this is noticing that we can use ϵ^{ij} to raise indices. Recall that in general the antifundamental $\bar{V}$ is equivalent to the $(N-1)$-th antisymmetric product of the fundamental representation, $\Lambda^{N-1}V$, a fact expressed by (4.9.31). In this case, $N = 2$, so $\bar{V}$ is equivalent to V.

More generally, if there is any pair of antisymmetric indices, sai i and j, in our tensor, we can use ϵ_{ij} to eliminate them: $\epsilon_{ij}M^{iji_3\ldots i_k} \equiv M^{i_3\ldots i_k}$. So we can limit ourselves to tensors which are completely symmetric, of the form (4.9.24). Alternatively, this follows from the general statement that the Young diagram of a representation of SU(N) *can be at most $N-1$ boxes deep.*

These representations are the same we have seen in example 4.21 and 4.23: the number of boxes is equal to $2l$.

Example 4.32. *Let us now consider* SU(3)*. In this case, the antifundamental is $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$. The adjoint 8 can be thought of as a matrix M that transforms as $M \mapsto UMU^{-1} = UMU^{\dagger}$. So it is a tensor $M^{i\bar{j}}$. It is traceless in the sense that $M^{i\bar{j}}\delta_{i\bar{j}} = 0$. Using the ϵ tensor as in (4.9.30), we can relate this to a tensor with three indices:*

$$M^{ijk} \equiv \epsilon^{jkl} M^{i}{}_{l} = \epsilon^{jkl} M^{i\bar{r}} \delta_{l\bar{r}} \; . \tag{4.9.33}$$

This tensor is antisymmetric in the last two indices; from the condition of tracelessness one can in fact see that it is $\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$.

So for example ([4.9.19]) reads

$$\ydiagram{1} \otimes \ydiagram{1,1} = \ydiagram{2,1} \oplus 1 \qquad (3 \otimes \bar{3} = 8 \oplus 1) \ . \tag{4.9.34}$$

On the other hand, ([4.9.21]) reads

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \qquad (3 \otimes 3 = 6 \oplus \bar{3}) \ . \tag{4.9.35}$$

4.10 The Lorentz and Poincaré groups

Special relativity teaches us that the "norm" in spacetime is proper time:

$$-\tau^2 = -t^2 + x^2 + y^2 + z^2 = x^\mu \eta_{\mu\nu} x^\nu \ , \tag{4.10.1}$$

where $x^\mu = (t, x, y, z)^t$, and

$$\eta \equiv \mathrm{diag}(-1, 1, 1, 1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{4.10.2}$$

When we consider $\mathbb{R}^4$ with this quadratic form, we call it $\mathbb{R}^{3,1}$ to emphasize its signature, and we call it *Minkowski space* (or *spacetime*). Consider a linear action on the coordinates,

$$x \to \Lambda x \ , \tag{4.10.3}$$

and impose that τ^2 should be invariant. This happens if $\Lambda^t \eta \Lambda = \eta$. So we define

$$\mathrm{O}(3,1) = \Big\{ \Lambda \in \mathrm{Mat}(4, \mathbb{R}) \mid \Lambda^t \eta \Lambda = \eta \Big\} \ . \tag{4.10.4}$$

This is the analogue in $\mathbb{R}^{3,1}$ of the orthogonal group. $\mathrm{O}(N)$ is the group of matrices that leave the quadratic form 1_N invariant, whereas $\mathrm{O}(3,1)$ is the group of matrices that leave invariant the quadratic form η in ([4.10.2]). It is also useful to introduce indices $\mu = 1, \ldots, 4$; $\eta_{\mu\nu}$ then has both indices down, $\Lambda^\mu{}_\nu$ has indices of both types, and $\Lambda^t \eta \Lambda = \eta$ reads

$$(\Lambda^t)_\mu{}^\nu \eta_{\nu\rho} \Lambda^\rho{}_\sigma = \eta_{\mu\sigma} \ . \tag{4.10.5}$$

Just like for $\mathrm{O}(N)$, we see that $\det(\Lambda)^2 = 1$; the component with $\det = -1$ is disconnected from the component with $\det = 1$. So we can define

$$\mathrm{SO}(3,1) = \Big\{ \Lambda \in \mathrm{O}(3,1) | \det(\Lambda) = 1 \Big\} \ . \tag{4.10.6}$$

However, in this case $\mathrm{SO}(3,1)$ is not connected, either. The reason can be seen by considering the component $\mu = \sigma = 0$ of (4.10.5):

$$({\Lambda^0}_0)^2 = 1 + {\Lambda^i}_0 {\Lambda^i}_0 \ . \tag{4.10.7}$$

This implies that ${\Lambda^0}_0$ is either ≥ 1 or ≤ 1. So even $\mathrm{SO}(3,1)$ splits in two components; the one which is a subgroup is

$$\mathrm{SO}^+(3,1) = \left\{ \Lambda \in \mathrm{SO}(3,1) \mid {\Lambda^0}_0 \geq 1 \right\} \ . \tag{4.10.8}$$

$\mathrm{O}(3,1)$ is called the *Lorentz group*, whereas (4.10.8) is called the *restricted* Lorentz group. An example of a Λ for which $\det(\Lambda) = -1$ is given by the parity transformation P : $x^0 \mapsto x^0$, $x^i \mapsto -x^i$; an example of a Λ for which ${\Lambda^0}_0 < 0$ is given by the time inversion $T : x^0 \mapsto -x^0$, $x^i \mapsto x^i$. One of the surprises of elementary particle physics is that P *and* T *are not symmetries of nature.* For this reason, in what follows we will restrict our attention to $\mathrm{SO}^+(3,1)$.

There is a close relationship with a group we know already:

Lemma 4.8. *There is a homomorphism:*

$$\mathrm{Sl}(2,\mathbb{C}) \xrightarrow{2:1} \mathrm{SO}^+(3,1) \ . \tag{4.10.9}$$

(At this point, this lemma is a bit of a curiosity. We will explain later that it does have a physical interpretation, however.)

Proof. To describe the homomorphism, consider first the following map from Minkowski space to the vector space of hermitian 2×2 matrices:

$$x^\mu \mapsto x^\mu \sigma_\mu \equiv x^0 1_2 + x^i \sigma_i \ . \tag{4.10.10}$$

It is easy to see that this map is invertible. Moreover, using the fact that for 2×2 matrices $\det(A) = \frac{1}{2}(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))$, we can see that

$$\det(x^\mu \sigma_\mu) = x^\mu \eta_{\mu\nu} x^\nu \ . \tag{4.10.11}$$

Now, if we have $S \in \mathrm{Sl}(2,\mathbb{C})$, the matrix $S(x^\mu \sigma_\mu)S^\dagger$ is also hermitian, so it can be written as $\tilde{x}^\mu \sigma_\mu$ for some other vector $\tilde{x}^\mu \in \mathbb{R}^{3,1}$. We can now define a matrix Λ_S via $\tilde{x}^\mu \equiv (\Lambda_S x)^\mu = \Lambda^\mu_{S\nu} x^\nu$, so that

$$S(x^\mu \sigma_\mu)S^\dagger \equiv \Lambda^\mu_{S\nu} x^\nu \sigma_\mu \ . \tag{4.10.12}$$

Since $\det(S x^\mu \sigma_\mu S^\dagger) = \det(x^\mu \sigma_\mu)$, we have that $x^\mu \eta_{\mu\nu} x^\nu = (\Lambda_S x)^\mu \eta_{\mu\nu} (\Lambda_S x)^\nu$; in other words, $\Lambda_S \in \mathrm{O}(3,1)$. This gives a map from $S \in \mathrm{Sl}(2,\mathbb{C})$ to $\Lambda_S \in \mathrm{O}(3,1)$. Since $\mathrm{Sl}(2,\mathbb{C})$ is connected, the image of this map should be connected too. So in fact $\Lambda_S \in \mathrm{SO}^+(3,1)$. Summing up, (4.10.12) is the map we anticipated in (4.10.9). (We will show later why it is two to one.) □

We now proceed to describe the Lie algebra of $\mathrm{SO}^+(3,1)$. This is done just like in section 4.3, by linearizing around the identity $\Lambda \sim 1 + \omega$. From $\Lambda^t \eta \Lambda = \eta$ we obtain $\omega^t \eta + \eta \omega = 0$; so

$$\mathrm{so}(3,1) = \left\{ \omega \in \mathrm{Mat}(4,\mathbb{R}) \mid \eta\omega = -(\eta\omega)^t \right\} . \tag{4.10.13}$$

Notice that then $\eta\omega$ is antisymmetric. In indices, ω has the same structure as Λ, with one index up and one down: $\omega^\mu{}_\nu$. So it wouldn't even make sense to ask that it should be antisymmetric. But $\eta\omega$ has two indices down. Since we use η to lower indices, we define

$$\omega_{\mu\nu} \equiv \eta_{\mu\rho} \omega^\rho{}_\nu \ , \tag{4.10.14}$$

and then we should have $\omega_{\mu\nu} = -\omega_{\nu\mu}$. It follows immediately that (4.10.13) has dimension 6.

We have defined the Lie algebra so(3,1) concretely from matrices, rather than abstractly from generators. We would now like to go back. To do that, we will pick an explicit basis, one which also makes good physical sense.

In the case of so(3), we just used matrices that had as many zeros as possible (see (4.5.21)); here, we are going to do the same. In fact, we might as well keep using the three $\rho_1(\ell_i)$, which were the generators of the rotations around each of the three axes in $\mathbb{R}^3$. We have to add rows of zeros, however, so as to add[13] the coordinate x^0:

$$\ell_1 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad \ell_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \qquad \ell_3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.10.15}$$

[13]We dropped the ρ_1 in (4.5.21), both because it's clumsy, and because in this case we have not given an earlier, abstract definition of the Lie algebra.

Since so$(3,1)$ has dimension 6, we still need three more generators. We will take

$$k_1 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad k_2 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad k_3 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{4.10.16}$$

The k_i in (4.10.16) are not antisymmetric! But their product with η is, as one can check. So the condition in (4.10.13) is satisfied. To interpret these k_i physically, we can compute one of their exponentials:

$$\exp(\lambda k_1) = \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) & 0 & 0 \\ \sinh(\lambda) & \cosh(\lambda) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.10.17}$$

This is a Lorentz transformation: it corresponds to changing frame in the direction x^1 by a velocity v defined by

$$\lambda = \operatorname{arccosh}(\gamma) = \operatorname{arctanh}(v/c) \ , \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \ . \tag{4.10.18}$$

The k_i are consequently called "boosts". The presence of these elements makes the Lorentz group *noncompact*: the Lorentz transformation (4.10.17) is not the identity for any value of γ (unlike a rotation about an axis in space, which becomes the identity for $\theta = 2\pi$).

Let us now turn to the Lie algebra structure in this basis. One way to do this is to collect the ℓ_i and k_i together:

$$j_{0i} \equiv k_i \ , \qquad j_{ij} = \epsilon_{ijk} \ell_k \ , \tag{4.10.19}$$

so that the components of $j_{\mu\nu}$ are given by

$$(j_{\mu\nu})^{\alpha\beta} \equiv -2\delta^\alpha_{[\mu}\delta^\beta_{\nu]} = -\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu \ , \tag{4.10.20}$$

so that now computing their algebra becomes a straightforward, if dull, exercise:

$$[j_{\mu\nu}, j^{\rho\sigma}] = 4\delta_{[\mu}{}^{[\rho} j_{\nu]}{}^{\sigma]} = \delta^\rho_\mu j_\nu{}^\sigma - \delta^\rho_\nu j_\mu{}^\sigma - \delta^\sigma_\mu j_\nu{}^\rho + \delta^\sigma_\nu j_\mu{}^\rho \ . \tag{4.10.21}$$

This does not look very informative, however. Alternatively, we can just compute the algebra of the ℓ_i and k_i directly:

$$[\ell_i, \ell_j] = \epsilon_{ijk} \ell_k \ , \tag{4.10.22a}$$

$$[\ell_i, k_j] = \epsilon_{ijk} k_k \ , \tag{4.10.22b}$$

$$[k_i, k_j] = -\epsilon_{ijk} \ell_k \ . \tag{4.10.22c}$$

(4.10.22a) is just the algebra of so(3); finding it inside so(3, 1) should not come as a surprise. (4.10.22b) tells us that the boosts k_i transform as a vector under this so(3) subalgebra.

(4.10.22) collectively have an interesting consequence:

Lemma 4.9. *The complexification* $\mathrm{so}(3,1)_{\mathbb{C}}$ *of the Lorentz algebra is isomorphic to two copies*[14] *of* $\mathrm{sl}(2,\mathbb{C})$*:*

$$\mathrm{so}(3,1)_{\mathbb{C}} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C}) \ . \tag{4.10.23}$$

Proof. It is enough to define

$$\ell_i^+ \equiv \frac{1}{2}(\ell_i + ik_i) \ , \qquad \ell_i^- \equiv \frac{1}{2}(\ell_i - ik_i) \ . \tag{4.10.24}$$

One then finds from (4.10.22):

$$[\ell_i^\pm, \ell_j^\pm] = \epsilon_{ijk}\ell_k^\pm \ , \qquad [\ell_i^+, \ell_j^-] = 0 \ . \tag{4.10.25}$$

We recognize two su(2) algebras; but since we have complexified the initial algebra so(3, 1), we are really allowed to take any complex combinations of the generators. So we really have two copies of $\mathrm{su}(2)_{\mathbb{C}} = \mathrm{sl}(2,\mathbb{C})$. □

In view of (4.10.9), this lemma is probably not shocking. In fact, one could have used (4.10.9) to derive it. We will use this lemma in section 4.10.2 to find all finite-dimensional representations of the Lorentz group.

So far we have looked at "rotations" in spacetime. We can also add translations, to obtain the so-called *Poincaré group*:

$$\mathrm{ISO}(3,1) = \left\{ \text{maps } x \mapsto \Lambda x + a \mid \Lambda \in \mathrm{SO}(3,1),\ a \in \mathbb{R}^{3,1} \right\} \ . \tag{4.10.26}$$

Again because the parity P and time inversion T transformations are not symmetries of nature, we will actually consider in what follows the *restricted Poincaré* group $\mathrm{ISO}^+(3,1)$, where $\Lambda \in \mathrm{SO}^+(3,1)$. Apart from this detail, the definition of (4.10.26) is very similar to the one of the Euclidean group we saw in (2.3.16); for example we have

$$\mathrm{ISO}^+(3,1) = \mathrm{SO}^+(3,1) \ltimes \mathbb{R}^{3,1} \ . \tag{4.10.27}$$

To study this group more explicitly, let's find a basis for its Lie algebra, and the commutation relations among its elements. As a basis, we have the generators of translations

[14] Here we mean the direct sum not only as a vector space, but even in the sense that generators of the two summands commute with each other: see definition 4.18.

P^μ, and the generators $J_{\mu\nu}$ of Lorentz transformations. We are using the upper-case J rather than the lower-case j in (4.10.19), to emphasize that we are looking at the abstract generator rather than at a particular representation. The Lorentz group is defined as a group of matrices, so there is no need to make that distinction in that case. But the Poincaré group doesn't really have a representation in terms of matrices, because of the generators P^μ. So we shouldn't play favorites with any particular representation; so P^μ and $J_{\mu\nu}$ are to be understood as abstract generators.

Now we look at their commutation relations. Translations commute among themselves, $[P^\mu, P_\nu] = 0$; the algebra of the $J_{\mu\nu}$ is just the same as the one (4.10.21) for the Lorentz group. So all we have to find is $[J_{\mu\nu}, P^\rho]$. We don't expect this to be zero, since the product in (4.10.27) is semidirect, rather than direct. Let us compute this commutation relation in a particular representation: namely, we represent the translation group as a derivative, as in example 4.18. Then $\rho(P_\mu) = \partial_\mu$. This can be extended to the Lorentz group by taking $\rho(J_{\mu\nu}) = -2x_{[\mu}\partial_{\nu]} = -x_\mu\partial_\nu + x_\nu\partial_\mu$. So now we can compute the algebra in this representation. We get

$$[J_{\mu\nu}, P^\rho] = 2\delta^\rho_{[\mu}P_{\nu]} \ . \tag{4.10.28}$$

Using (4.10.20) can rewrite this in terms of the "fundamental" representation $j_{\mu\nu}$ of the Lorentz group:

$$[J_{\mu\nu}, P^\rho] = -(j_{\mu\nu})^\rho{}_\sigma P^\sigma \ . \tag{4.10.29}$$

This means that P^ρ transforms as a vector under $J_{\mu\nu}$, which makes good physical sense.

Now that we have described the Lorentz and Poincaré groups, we will turn to describing some of their representations.

Let us introduce such representations from a physical point of view. First, let us require that we have a quantum relativistic theory. Since we have a quantum theory, there should be a Hilbert space $\mathcal{H}$. The states in $\mathcal{H}$ should transform both under translations and rotations. Moreover, the generators of these symmetries should be represented by hermitian operators. So:

$$\mathcal{H} \ = \ \left\{ \begin{array}{c} \text{unitary representation} \\ \text{of the Poincaré group} \end{array} \right\} \ . \tag{4.10.30}$$

Since the Poincaré group is noncompact, theorem 4.7 tells us that a unitary representation will be infinite-dimensional. This sounds scary, but we will see that this is actually physically sensible: it is a consequence of the infinitely many possible values for the momentum quantum number.

Today we know that relativistic quantum theories are theories of "fields". A field assembles together creators and annihilators for many different quantum states. Translations do act on fields, but they do so as derivatives ∂_μ. So the interesting part of their transformation law is under the Lorentz group. For the fields to be manageable, we want them to have finitely many components, so:

$$\text{field} \;=\; \left\{ \begin{array}{c} \text{finite-dim. representation} \\ \text{of the Lorentz group} \end{array} \right\} . \tag{4.10.31}$$

The Lorentz group is also non-compact, so theorem 4.7 again applies, and it tells us that a finite-dimensional representation cannot be unitary. But this is not an issue. Getting a bit ahead of ourselves, the representation implicit in our description of $\mathrm{SO}(3,1)$ in terms of matrices is the "vector", or "fundamental" representation. One such a field is the electro-magnetic potential A^μ, which transforms just like a coordinate:

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu . \tag{4.10.32}$$

This representation is not unitary: for example, $\Lambda^\mu{}_\nu$ in (4.10.17) is not a unitary matrix. But A^μ itself is not a state —- it is a field. So there is no need to worry.

Mathematicians have also looked at unitary representations of the Lorentz group, but don't be fooled — they are no use to us, at least not at this stage.[15]

In the following two subsections, we will look in turn at the unitary representations of the Poincaré group, and then at the finite-dimensional representations of the Lorentz group.

4.10.1 Unitary representations of the Poincaré group

In this section, we will actually deal with *one-particle* states, which are irreducible representations. These were studied by Wigner; a good modern reference is [5, Vol. 1, 2.5].

We need a bit of notation: an element of the Poincaré group consists in a pair (Λ, a), where $\Lambda \in \mathrm{O}(3,1)$ and a is a translation. We will denote the operator representing this element by

$$U(\Lambda, a) . \tag{4.10.33}$$

This operator will act on states, for which we will use Dirac's ket notation. Since the momenta P_μ all commute with each other, they can be simultaneously diagonalized. So it is natural to work in this basis:

$$P_\mu |p, \sigma\rangle = i p_\mu |p, \sigma\rangle \tag{4.10.34}$$

[15] Amusingly, the group $\mathrm{SO}(4,1)$ is the Lorentz group for $\mathbb{R}^{1,4}$, but it is also the analogue of the Poincaré group for the so-called de Sitter space, which approximates a universe with a positive cosmological constant; unitary representations of this group are then relevant again for physics.

where σ denotes collectively any additional quantum numbers we might need later. We already see that the representation is infinite-dimensional, because of the continuous labels p_μ. (The unpleasant i is due to the fact that our generators are antihermitian, so that they close under commutator.)

At this point, we already know how to represent translations $U(0,a)$: by exponentiating (4.10.34), we get $U(0,a)|p,\sigma\rangle = e^{ia_\mu p^\mu}|p,\sigma\rangle$. All we have to do is represent Lorentz transformations $U(\Lambda) \equiv U(\Lambda, 0)$. First of all we notice:

Lemma 4.10. *A Lorentz transformation $U(\Lambda) \equiv U(\Lambda, 0)$ takes a state of momentum p into a state of momentum Λp.*

Proof. Let us first compute the adjoint action of $U(\Lambda)$ on the element P^μ of the Lie algebra. To do so, write $U(\Lambda) = e^{\omega_{\mu\nu}J^{\mu\nu}}$; we can now use the usual Hadamard trick (4.4.6) and the commutation relation (4.10.29) to get

$$e^{-\omega_{\mu\nu}J^{\mu\nu}} P^\rho e^{\omega_{\mu\nu}J^{\mu\nu}} = [\exp(\omega_{\mu\nu}j^{\mu\nu})]^\rho{}_\sigma P^\sigma \ ; \tag{4.10.35}$$

I am not showing you the steps in detail this time, because this computation is very similar to (4.7.6). This means

$$U(\Lambda^{-1})P^\rho U(\Lambda) = \Lambda^\rho{}_\sigma P^\sigma \ . \tag{4.10.36}$$

This is the finite version of (4.10.29), in our representation $U(\Lambda, a)$. Now we have:

$$P^\rho(U(\Lambda)|p,\sigma\rangle) = U(\Lambda)U(\Lambda^{-1})P^\rho U(\Lambda)|p,\sigma\rangle = U(\Lambda)(\Lambda^\rho{}_\sigma P^\sigma)|p,\sigma\rangle = i(\Lambda p)^\rho U(\Lambda)|p,\sigma\rangle \tag{4.10.37}$$

which is just what we meant to show. □

This would almost seem to solve our problem: we might just choose a certain fixed k, and define

$$|\Lambda k, \sigma\rangle \equiv U(\Lambda)|k,\sigma\rangle \ . \tag{4.10.38}$$

This is a good idea; the only problem is that, whatever k we choose, there are some Λ that keep it invariant.

Definition 4.25. *Given a group G and a representation ρ on V, the* little group *(or* isotropy group*) of a $v \in V$ is the subgroup G_v of elements that leave v invariant:*

$$\mathrm{ISO}_G(v) = \{g \in G \mid \rho(g)v = v\} \ . \tag{4.10.39}$$

In the case of the Lorentz group, the little group of any p_μ is non-zero. So (4.10.38) is not enough to represent the Lorentz group. But once we solve this problem with the little group, we are done!

Theorem 4.11. *A representation of the Poincaré group is determined by a representation of the little group* ISO(k) *for a fixed* k.

Proof. Choose k and keep it fixed in this construction. Then, for every other momentum p, choose a Lorentz transformation Λ_p such that

$$p = \Lambda_p k \ . \tag{4.10.40}$$

(Notice that this is not unique, precisely because of the little group: for any $\Lambda \in \mathrm{ISO}(k)$, $\Lambda_p \Lambda k = \Lambda_p k = p$. So we really need to choose a concrete Λ_p for any p). Let's make (4.10.38) more precise now by writing

$$|p, \sigma\rangle \equiv U(\Lambda_p)|k, \sigma\rangle \ . \tag{4.10.41}$$

Again, this makes sense because the state on the right has momentum $\Lambda_p k = p$, thanks to lemma 4.10.

Now, suppose you have a representation of the little group ISO(k); call $U_{\sigma\sigma'}$ the corresponding matrices

$$U(\Lambda)|k, \sigma\rangle \equiv U_{\sigma\sigma'}(\Lambda)|k, \sigma'\rangle \ , \qquad \Lambda \in \mathrm{ISO}(k) \ . \tag{4.10.42}$$

We have to show that we know how to represent any element Λ of the Lorentz group. We need to show that $U(\Lambda)|p, \sigma\rangle$ can be expressed in terms of the representation $U_{\sigma\sigma'}$ of the little group. First, notice that, for any $\Lambda \in \mathrm{SO}^+(3, 1)$,

$$(\Lambda^{-1}_{\Lambda p} \, \Lambda \, \Lambda_p) \, k = \Lambda^{-1}_{\Lambda p} \, \Lambda \, p = k \ ; \qquad \Rightarrow \ \Lambda^{-1}_{\Lambda p} \, \Lambda \, \Lambda_p \in \mathrm{ISO}(k) \ . \tag{4.10.43}$$

Now we evaluate

$$\begin{aligned} U(\Lambda)|p, \sigma\rangle &= U(\Lambda_{\Lambda p})U(\Lambda^{-1}_{\Lambda p})U(\Lambda)U(\Lambda_p)|k, \sigma\rangle = U(\Lambda_{\Lambda p})U(\Lambda^{-1}_{\Lambda p}\Lambda\Lambda_p)|k, \sigma\rangle \\ &= U(\Lambda_{\Lambda p})U_{\sigma\sigma'}(\Lambda^{-1}_{\Lambda p}\Lambda\Lambda_p)|k, \sigma'\rangle = U_{\sigma\sigma'}(\Lambda^{-1}_{\Lambda p}\Lambda\Lambda_p)|\Lambda p, \sigma'\rangle \ . \end{aligned} \tag{4.10.44}$$

This gives the desired representation of Λ on any state $|p, \sigma\rangle$. □

Nobody says (4.10.44) is particularly pretty. However, we have now reduced the classification of representations of the Poincaré group to representations of ISO(k). So now we need to know what that group is.

There are essentially[16] two cases to be considered: $k^\mu \eta_{\mu\nu} k^\nu \equiv k^2 < 0$ (massive particles) and $k^2 = 0$ (massless particles).

In the massive case, we can take $k = (1, 0, 0, 0)$, which is indeed such that $k^2 < 0$. Then it is immediate and intuitive to see that the elements ISO(k) are space rotations: a Lorentz transformation such as (4.10.17) would not leave k invariant. So we have

$$\mathrm{ISO}(k) = \mathrm{SO}(3) \qquad (k^2 < 0) \ . \tag{4.10.45}$$

This means that massive particles are classified by representations of SO(3). Fortunately, we know these very well: they are classified by *spin*. So this is what the label σ in (4.10.44) signifies.

The massless case is less straightforward. We can take for example $k = (1, 1, 0, 0)$; but understanding its little group is now less intuitive. Let's do it at the infinitesimal level. If $\Lambda \sim 1 + \omega$ at first order, then $\Lambda k = k$ implies $\omega k = 0$. So we have to find elements of the Lie algebra so$(3,1)$ that leave k invariant. Using the explicit expressions (4.10.15) and (4.10.16), we find that the most general such element is of the form

$$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & a & b \\ a & -a & 0 & -\theta_1 \\ b & -b & \theta_1 & 0 \end{pmatrix} = \theta_1 \ell_1 + a(k_2 - \ell_3) + b(k_3 + \ell_2) \ . \tag{4.10.46}$$

Using (4.10.22), we compute the non-zero commutation relations

$$[\ell_1, k_2 - \ell_3] = k_3 + \ell_2 \ , \qquad [\ell_1, k_3 + \ell_2] = -(k_2 - \ell_3) \tag{4.10.47}$$

One can see (**exercise!**) that this is isomorphic to the Lie algebra of the Euclidean group $E(2)$ (see example 2.14), namely the group of rotations and translations in two dimensions; ℓ_1 being the rotation, $k_2 - \ell_3$ and $k_3 + \ell_2$ playing the role of "translations". The group $E(2)$ is non-compact. Fortunately, theorem 4.7 does not apply, because the Killing form is actually negative-semidefinite, and so the algebra is not non-compact. It is still true, however, that most unitary representations of $E(2)$ are infinite-dimensional; they would require a new continuous label. Such a quantum number is not observed. The only way to get rid of it is to represent the generators $k_2 - \ell_3$ and $k_3 + \ell_2$ trivially, and to look at representations of the remaining generator ℓ_1. The corresponding Lie group

[16] Since we are considering the restricted Poincaré group, which doesn't include the time inversion T, we should also distinguish $k_0 > 0$ and $k_0 < 0$; we will however impose $k_0 > 0$. We are also ignoring the unphysical cases where the momentum is spacelike, $k^2 > 0$, and where it is identically zero, $k_\mu = 0$.

is U(1); its representations are characterized by an integer, as we saw in (4.5.18). This integer is called *helicity* and is the analogue of spin for massless particles. Of course we couldn't encounter this concept in non-relativistic quantum mechanics!

4.10.2 Finite-dimensional representations of the Lorentz group

These representations are relevant to classify the possible fields in a quantum field theory.

We will look for representations of the Lie algebra so$(3,1)$. Just like we did for su(2), we can first look for representations of the complexified algebra so$(3,1)_{\mathbb{C}}$; each of these can then be restricted to a representation of so$(3,1)$, simply by taking the coefficients of the generators to be real rather than complex. Notice that, if we were to impose that the representation be hermitian (namely that $i\rho(\ell_i)$, $i\rho(k_i)$ be hermitian matrices), we would be sure to fail, because of theorem 4.7. But, as noticed under (4.10.31), there is no need to impose hermiticity.

Fortunately, we know from lemma 4.9 that the complexification so$(3,1)_{\mathbb{C}} \cong$ sl$(2,\mathbb{C}) \oplus$ sl$(2,\mathbb{C})$. We have already classified all finite-dimensional representations of sl$(2,\mathbb{C})$ in example 4.21: they are characterized by one integer l. Since we have two copies of sl$(2,\mathbb{C})$, representations of so$(3,1)$ are characterized by two half-integers (l_1, l_2). To summarize what we just concluded:

Theorem 4.12. *Finite-dimensional representations of* so$(3,1)$ *are characterized by two half-integers* (l_1, l_2).

This might look disconcerting at first: are there "two spins"? in a sense, yes. To explain this, we need to generalize the concept of spinor to any dimension. This requires, in turn, that we generalize Pauli matrices, since they appear in the spin representation (4.6.7). As we will now see, the crucial property is the anticommutator (4.6.5):

Definition 4.26. *The* Clifford algebra $Cl(3,1)$ *is the algebra spanned by elements* γ_μ*, such that*

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \ . \tag{4.10.48}$$

It can be shown that such an algebra can be represented by 4×4 matrices, and not any smaller. Moreover, all representations are equivalent up to a change of basis. An example of four matrices γ_μ that satisfy (4.10.48) is given by

$$\gamma^0 = i\sigma^1 \otimes 1_2 = i \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} , \qquad \gamma^i = \sigma^2 \otimes \sigma^i = i \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} . \tag{4.10.49}$$

We used both a tensor product notation (in which the matrices act on $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^{2\times 2} = \mathbb{C}^4$), and a block notation. The first is very convenient to compute products: for example,

$$\gamma^0 \gamma^i = (i\sigma^1 \sigma^2) \otimes (1_2 \sigma^i) = (-i\sigma^2 \sigma^1) \otimes (\sigma^i 1_2) = -\gamma^i \gamma^0 \ , \tag{4.10.50}$$

which is the $\mu = 0$, $\nu = i$ component of (4.10.48). The other components can be checked similarly.

Now, the crucial observation is that matrices of the form

$$\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu] = \gamma_{[\mu}\gamma_{\nu]} \tag{4.10.51}$$

satisfy, thanks to (4.10.48),

$$[\gamma_{\mu\nu}, \gamma^{\rho\sigma}] = -8\, \delta_{[\mu}{}^{[\rho} \gamma_{\nu]}{}^{\sigma]} \ . \tag{4.10.52}$$

This is very similar to the algebra of the Lorentz group in (4.10.21): so similar that

$$\rho_s(J_{\mu\nu}) = -\frac{1}{2}\gamma_{\mu\nu} \tag{4.10.53}$$

defines a representation of the Lie algebra so$(3,1)$, which we will call the *spin* representation. The elements of the vector space on which this representation acts will be called *spinors*. We can promote it to a representation of SO$(3,1)$ by the usual matrix exponential:

$$\rho_S(e^{\omega_{\mu\nu} J^{\mu\nu}}) = \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right] \ . \tag{4.10.54}$$

We can actually define the group

$$\mathrm{Spin}(3,1) \equiv \left\{ \exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right] , \ \omega_{\mu\nu} = -\omega_{\nu\mu} \right\} \ . \tag{4.10.55}$$

With similar steps as those that took us to (4.6.21), we can find

$$\mathrm{Spin}(3,1) \xrightarrow{2:1} \mathrm{SO}^+(3,1) \ . \tag{4.10.56}$$

We will see shortly that this homomorphism is related to the one we saw in (4.10.9).

The spin representation (4.10.53) is not actually irreducible. To see this, define

$$\gamma \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \ \Rightarrow \ \{\gamma, \gamma^\mu\} = 0 \quad \forall \mu \ . \tag{4.10.57}$$

You can also check $\gamma^2 = 1$; so

$$P_\pm = \frac{1}{2}(1 + \gamma) \tag{4.10.58}$$

are projectors. The subspaces $S_\pm \equiv \mathrm{im}(P_\pm)$ are invariant under ρ_s. To see this, consider $\eta_\pm \in S_\pm$. In other words, $\gamma\eta_\pm = \pm\eta_\pm$. Then the transformed spinor $\exp[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}]\eta_\pm$ are still in $S_\pm$:

$$\gamma \exp[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}]\eta_\pm = \exp[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}]\gamma\eta_\pm = \pm \exp[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}]\eta_\pm \ , \tag{4.10.59}$$

where we have used that $[\gamma, \gamma^{\mu\nu}] = 0$ (see (4.10.57)). We will call $S_\pm$ the space of spinors with positive or negative *chirality*.

We can see this more explicitly in our basis (4.10.49). We have $\gamma = \sigma_3 \otimes 1$; so the subspaces $S_\pm$ are simply given by spinors with the second two entries equal to zero, or the first two entries equal to zero, respectively. Now, since the γ_μ in our basis are anti-block-diagonal, the $\gamma_{\mu\nu}$ will be block-diagonal, and hence will leave $S_\pm$ invariant. More precisely, let us rewrite

$$\omega_{\mu\nu}J^{\mu\nu} = 2\omega_{0i}J^{0i} + \omega_{ij}J^{ij} = -2\omega_{0i}k^i + \omega_{ij}\epsilon^{ijk}\ell_k = \ell_i^+\omega_i^+ + \ell_i^-\omega_i^- \ , \tag{4.10.60}$$

where we have defined

$$\omega_i^\pm = \frac{1}{\sqrt{2}}(\epsilon_{ijk}\omega^{jk} \pm 2i\omega_{0i}) \ . \tag{4.10.61}$$

Now use the basis (4.10.49) to compute $\gamma^{0i} = -\sigma^3 \otimes \sigma^i$ and $\gamma^{ij} = 1_2 \otimes i\epsilon^{ijk}\sigma_k$. Using this, we get

$$\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu} = \frac{i}{\sqrt{2}}\begin{pmatrix} \omega_i^+\sigma^i & 0 \\ 0 & \omega_i^-\sigma^i \end{pmatrix} \ . \tag{4.10.62}$$

So we see that the two spaces $S_\pm$ have a clear interpretation in terms of the two copies of $\mathrm{sl}(2,\mathbb{C})$: S_+ transforms under ℓ_i^+, but not under ℓ_i^-; since it is two-dimensional, it corresponds to

$$(l_+, l_-) = \left(\frac{1}{2}, 0\right) \ . \tag{4.10.63}$$

On the other hand, S_- is the representation $(0, 1/2)$.

Incidentally, if we exponentiate (4.10.62), since $(\omega_i^+)^* = \omega_i^-$ we get a matrix of the form

$$\exp\left[-\frac{1}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right] = \begin{pmatrix} S & 0 \\ 0 & (S^{-1})^\dagger \end{pmatrix} \ , \qquad S \in \mathrm{Sl}(2,\mathbb{C}) \ . \tag{4.10.64}$$

If we now compute the adjoint action of this on the matrix

$$x^\mu\gamma_\mu = -ix^0\sigma^1 \otimes 1_2 + \sigma^2 \otimes x^i\sigma_i = -i\begin{pmatrix} 0 & x^0 + x^i\sigma_i \\ -x^0 + x^i\sigma_i & 0 \end{pmatrix} \ , \tag{4.10.65}$$

the upper-right block gives us exactly the action (4.10.12). So we see now that the homomorphism (4.10.9) is the same as the homomorphism (4.10.56); in fact,

$$Spin(3,1) \cong \mathrm{Sl}(2,\mathbb{C}) \ . \tag{4.10.66}$$

Some representations can be understood without all this technology. A useful piece of information comes from considering how a representation of $\mathrm{SO}^+(3,1)$ decomposes under its subgroup of rotations, SO(3). We know that $\ell_i = \ell_i^+ + \ell_i^-$. This tells us that, under the $\mathrm{SO}(3) < \mathrm{SO}^+(3,1)$ subgroup, a representation (l_+, l_-) transforms in the $l_+ \otimes l_-$. We know that this can be decomposed as a direct sum of irreducible representations with spins between $l_+ + l_-$ and $|l_+ - l_-|$.

Let us consider for example the vector representation, the representation (of dimension four) implicit in our definition of the Lorentz group. It is four-dimensional, and irreducible. We know that a representation (l_+, l_-) has dimension $(2l_+ + 1)(2l_- + 1)$, so that leaves us three possibilities: $(1/2, 1/2)$, $(3/2, 0)$, or $(0, 3/2)$. Under the rotation subgroup, the vector representation decomposes as a direct sum of a singlet (the time component), and a triplet (the three space components). So the correct answer for the vector representation is:

$$\left(\frac{1}{2}, \frac{1}{2}\right) \qquad \text{(vector representation)} \ . \tag{4.10.67}$$

Indeed, for SO(3) we have $1/2 \otimes 1/2 = 0 \oplus 1$, which reproduces the expected decomposition under $\mathrm{SO}(3) < \mathrm{SO}(3,1)$.

To see (4.10.67) more directly, we can introduce indices for $S_\pm$. A spinor in S_+ will be denoted by η_α, $\alpha = 1,2$; a spinor in S_- will be denoted by $\eta_{\dot\alpha}$, $\dot\alpha = 1,2$. Notice that each γ^μ changes chirality:

$$\eta_+ \in S_+ \ \Rightarrow \ \gamma\eta_+ = \eta_+ \ \Rightarrow \ \gamma(\gamma^\mu \eta_+) = -\gamma^\mu \gamma \eta_+ = -\gamma^\mu \eta_+ \ \Rightarrow \ \gamma^\mu \eta_+ \in S_- \ . \tag{4.10.68}$$

So γ^μ has components $\gamma^\mu_{\alpha\dot\beta}$, and $\gamma^\mu_{\dot\alpha\beta}$, but no $\gamma^\mu_{\alpha\beta}$ nor $\gamma^\mu_{\dot\alpha\dot\beta}$. Now, given two spinors of opposite chirality η_α and $\epsilon_{\dot\alpha}$, we can construct a vector

$$\eta_\alpha \gamma_\mu^{\alpha\dot\beta} \epsilon_{\dot\beta} \ . \tag{4.10.69}$$

This shows that a vector representation can be seen as the tensor product of a $(1/2, 0)$ and of a $(0, 1/2)$.

In a way, this teaches us that spinor indices α and $\dot\alpha$ are more fundamental than vector indices μ, since the latter type can be "created" from two of the former, as in (4.10.69). The most general representation can then be seen as a "tensor"

$$S_{\alpha_1 \ldots \alpha_{l_+} \dot\beta_1 \ldots \dot\beta_{l_-}} \tag{4.10.70}$$

which is completely symmetric in the α_i and $\dot{\beta}_j$ separately.

5 Classification of Lie algebras

We have seen a fair number of Lie algebras, and perhaps you might feel you had enough of them already. It is true that you have already seen most of the Lie algebras you will really need in your life. Why then waste time studying a general classification? First of all, sometimes we want to know if a Lie algebra with certain features exists or not; it is nice in those cases to have a list of possibilities. If it were for this, however, we could skip directly to the list in section 5.4. The second reason to study a classification is that it gives us a deeper understanding of the structure of the algebras — for example, it allows us to classify representations. (So far, we have only classified representations for su(2) and its various relatives.)

Actually, a general classification of Lie algebras is out of the question — there are too many! however, the ones of most interest to physics are the so-called *semi-simple* algebras, which are the ones we will study from section 5.2 on. We first have to explain what they are, and why they are more interesting (for physics) than all the others.

5.1 Generalities

We will deal from now on with *complex* Lie algebras.

Let us start by defining

Definition 5.1. *The* derived algebra

$$\mathfrak{g}^1 \equiv [\mathfrak{g}, \mathfrak{g}] = \Big\{ z \in \mathfrak{g} \mid \exists\, x, y \in \mathfrak{g},\ [x, y] = z \Big\} \tag{5.1.1}$$

of a Lie algebra $\mathfrak{g}$ is its subalgebra of all elements that can be obtained as $[x, y]$, for any $x, y \in \mathfrak{g}$. In the same way, we can consider iteratively $\mathfrak{g}^k \equiv [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$.

Example 5.1.

- *Consider the three-dimensional "Heisenberg algebra" given in (4.5.10), whose only non-zero commutator is $[e_1, e_2] = e_3$. In this case, $\mathfrak{g}^1 = \langle e_3 \rangle$. Obviously then $\mathfrak{g}^2 = \{0\}$.*
- *For su(2), on the other hand, all elements can be obtained as commutators of two other elements. So $\mathfrak{g}^1 = \mathfrak{g}$.*

These two examples are extreme, in that in one case $\mathfrak{g}^k$ eventually becomes zero, and in the second case $\mathfrak{g}^k = \mathfrak{g}$ for all k. We now formalize this difference in general.

Definition 5.2. *A Lie algebra* $\mathfrak{g}$ *is called* solvable *if* $\mathfrak{g}^k = \{0\}$ *for some* k.

This extends the example of the Heisenberg algebra. To get an idea of what the general solvable algebra looks like, consider that:

Theorem 5.1. ***(Lie.)*** *A solvable Lie algebra can always be represented as a subalgebra of the Lie algebra of upper-triangular matrices (including the diagonal).*[17]

To generalize the other extreme, illustrated by su(2) in example 5.1, we define:

Definition 5.3. *A Lie algebra* $\mathfrak{a}$ *is called* abelian *if* $[x, y] = 0$, $\forall x, y \in \mathfrak{a}$.

Definition 5.4. *A Lie algebra is called* semisimple *if it has no non-trivial*[18] *abelian ideals. It is called* simple *if it has no non-trivial ideals at all, and if it is non-abelian.*

In fact:

Lemma 5.2. *A semisimple Lie algebra is a direct sum of simple Lie algebras.*[19]

It is easy to check (**exercise**) that su(2) is simple. More generally, we observe that clearly:

Lemma 5.3. *The derived algebra* $\mathfrak{g}^1$ *is an ideal (or an invariant subalgebra: see definition 4.18). In other words,* $[\mathfrak{g}, \mathfrak{g}^1] = \mathfrak{g}^1$.

So whenever $\mathfrak{g}^1$ is strictly contained in $\mathfrak{g}$, $\mathfrak{g}$ is not semisimple. In other words, if $\mathfrak{g}$ is semisimple, $\mathfrak{g}^1 = \mathfrak{g}$. This is exactly what we observed for su(2) in example 5.1.

The most general Lie algebra is a mix of the two types of algebras we just defined:

Theorem 5.4. *Every complex Lie algebra* $\mathfrak{g}$ *can be written as a* vector space *direct sum* $\mathfrak{g}_{\mathrm{solv}} \oplus \mathfrak{g}_{\mathrm{ss}}$. *Unlike for lemma 5.2, this is not a Lie algebra vector sum: the two summands do not commute with each other. However,* $\mathfrak{g}_{\mathrm{ss}}$ *is an ideal.*

$\mathfrak{g}_{solv}$, also called the "radical", is constructed as the sum of all the ideals of $\mathfrak{g}$.

There is a nice criterion to decide whether an algebra is semisimple, solvable or something in between.

[17] One can also define *nilpotent* Lie algebras as those for which the series $\mathfrak{g}_k \equiv [g, \mathfrak{g}_{k-1}]$ gets to zero for some k. One can show that these algebras can always be represented as a subalgebra of the Lie algebra of upper-triangular matrices, diagonal *excluded* — which explains their name.

[18] The "trivial" ideals, which every Lie algebra $\mathfrak{g}$ has, are $\{0\}$ and all of $\mathfrak{g}$.

[19] We mean this direct sum in the sense of definition 4.18, not only as a direct sum of vector spaces. To prove this lemma, we would need to apply theorem 5.21, that we will see later, to the adjoint representation.

Theorem 5.5. *If the Killing form $k(x, y) = 0$ for any $x \in \mathfrak{g}$, and any $y \in \mathfrak{g}^1$, then $\mathfrak{g}$ is solvable.*

So the Killing form in this case is not of maximal rank. On the other hand:

Theorem 5.6. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form k is non-degenerate.*

We saw indeed in example 4.26 that the Killing form for su(2) is proportional to the identity. For the three-dimensional Heisenberg algebra, on the other hand, you can check (**exercise**) that the Killing form is actually zero.

Now let us consider the non-abelian Yang–Mills action:

$$S_{\rm YM} = -\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\int F^i_{\mu\nu}F^{j\,\mu\nu}\mathrm{Tr}(T_iT_j) = -\int F^i_{\mu\nu}F^{j\,\mu\nu}k_{ij}\ . \tag{5.1.2}$$

In the last step, we have used the fact that $F_{\mu\nu}$ is supposed to be in the adjoint representation; so $\mathrm{Tr}(T_iT_j) = k_{ij}$ is nothing but the Killing form. We don't want k_{ij} to be degenerate, because there would be some degree of freedom without kinetic term. So the Lie algebras we should use for non-abelian Yang–Mills should be semisimple, because of theorem 5.6. From now on, then, we will restrict our attention to semisimple Lie algebras.

In fact, we also want k_{ij} to be negative definite: otherwise some degrees of freedom will have a wrong-sign kinetic term in their action. Theorem 4.6 then tells us that the gauge group should also be compact, which is what we do in particle physics. Their Lie algebras are real. Actually, it is easier to study complex Lie algebras, which is what we will do in the following subsections. Fortunately, we can keep in mind that:

Lemma 5.7. *Every semisimple Lie algebra has a compact real subalgebra.*

The invertibility of the Killing form has also a more technical consequence. We concluded in (4.8.27) that $f_{ijk} \equiv k_{il}f^l{}_{jk}$ are antisymmetric in all indices; but in general it is not possible to reconstruct the $f^i{}_{jk}$ from the f_{ijk}. For semisimple algebras, however, we can do so simply by writing $f^i{}_{jk} = k^{il}f_{ljk}$, where k^{ij} is the inverse of k_{ij}. In fact, by appropriate rescalings, one can find a basis where $k_{ij} = \delta_{ij}$; then there is no difference between upper and lower indices, and we can actually write $f_{ijk} = f^i{}_{jk}$. In other words:

Lemma 5.8. *For semisimple Lie algebras, there exists a basis in which the structure constants are completely antisymmetric.*

This generalizes the usual basis ℓ_i for su(2), where the structure constants are ϵ_{ijk}, which is indeed a completely antisymmetric tensor.

5.2 Semisimple Lie algebras: roots

We will now classify all semisimple Lie algebras. Remarkably, this will result in a very compact list of possibilities, summarized in the diagrams in figure 12.

The general idea of the classification is to try to generalize the basis H, X, Y in (4.8.2) of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$, and the corresponding commutation rules (4.8.3). First we want to generalize H:

Definition 5.5. *A subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is called* Cartan subalgebra *if it is abelian, diagonalizable (namely, for any* $H \in \mathfrak{h}$*,* $\mathrm{ad_H}$ *is diagonalizable) and maximal (namely, there is no larger abelian subalgebra* $\mathfrak{h} \subset \mathfrak{h}' \subset \mathfrak{g}$*).*

There also exists a more general definition, which also applies to non-semisimple algebras, but we are not going to need it.

A Cartan subalgebra **always exists**. (One can show this by introducing a generalization of the Jordan decomposition, which does not use the matrix representation of the algebra.) Moreover, any two Cartan subalgebras are related to one another by a change of basis. The dimension of a Cartan subalgebra of $\mathfrak{g}$ is called the **rank** r of $\mathfrak{g}$.

Let us introduce a basis $\{H_a\}_{a=1}^r$ in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. As we mentioned, these generalize H of $\mathrm{sl}(2,\mathbb{C})$. We will now try to generalize X and Y of $\mathrm{sl}(2,\mathbb{C})$. Since the $\mathrm{ad_{H_a}}$ are diagonalizable and commute with each other, they are simultaneously diagonalizable. Hence we can write $\mathfrak{g}$ as a sum of eigenspaces of the ad_{H_a}. Given an eigenvector x_α, we can write:

$$\mathrm{ad}_{H_a} x_\alpha = [H_a, x_\alpha] = \alpha_a x_\alpha \ . \tag{5.2.1}$$

The α_a can be thought of as the components of a vector $\alpha \in \mathbb{C}^r$, which we will call *root*. Let us also call L_α an eigenspace: in other words, the set of all the x_α which satisfy (5.2.1) for a given choice of α_a. We can also think of $\mathfrak{h}$ as an eigenspace: it is the eigenspace for which α is the zero vector in $\mathbb{C}^r$:

$$\mathfrak{h} = L_{\underline{0}} \ . \tag{5.2.2}$$

We will call Φ the set of all the roots. Summarizing, we have decomposed

$$\mathfrak{g} = \oplus_{\alpha\in\Phi} L_\alpha \ . \tag{5.2.3}$$

Example 5.2.

- *For* $\mathrm{sl}(2,\mathbb{C})$*, the Cartan subalgebra can be taken to be the one-dimensional space* $\langle H\rangle$*. The rank is* $r=1$*. From (4.8.3) we see that both* X *and* Y *are eigenvectors of* ad_H*; so they are examples of* x_α*. The roots are* ± 1*, thought of as vectors in* $\mathbb{C}^r = \mathbb{C}$*.*

- *For* $\mathfrak{g} = \mathrm{sl}(N, \mathbb{C})$*, a possible choice of Cartan subalgebra is the space of diagonal matrices in* $\mathrm{sl}(N, \mathbb{C})$*:* $\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, \ldots, a_N) \mid \sum_{i=1}^{N} a_i = 0\}$*. So the rank in this case is* $N - 1$*.*

Let us now derive a few general properties of this decomposition.

First of all, notice that the set of all roots generates all of $\mathbb{C}^r$:

$$\mathrm{Span}_{\mathbb{C}}(\Phi) = \mathbb{C}^r \ . \tag{5.2.4}$$

If this were not true, we would have that $\exists\, H \in \mathfrak{h}$ such that $[H, x_\alpha] = 0$, $\forall \alpha$. But such a H would then commute with all of $\mathfrak{g}$: in particular, it would generate a non-trivial abelian ideal, which is not possible when $\mathfrak{g}$ is semisimple.

Next, we remark that:

Lemma 5.9. *If* α *and* β *are two roots:*

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta} \tag{5.2.5}$$

Proof.

$$[H_a, [x_\alpha, x_\beta]] = [[H_a, x_\alpha], x_\beta] + [x_\alpha, [H_a, x_\beta]] = (\alpha_a + \beta_a)[x_\alpha, x_\beta] \ . \tag{5.2.6}$$

Notice that this formula is also true when $\alpha + \beta$ *is not a root*! in that case, however, $L_{\alpha+\beta} = \emptyset$, and hence $[x_\alpha, x_\beta] = 0$. □

Lemma 5.10. *If* $\alpha + \beta \neq 0$*,* L_α *and* L_β *are orthogonal with respect to the Killing form (4.8.24).*

Proof. Because of the invariance of the Killing form (see (4.8.26)):

$$\alpha_a k(x_\alpha, x_\beta) = k([H_a, x_\alpha], x_\beta) = -k(x_\alpha, [H_a, x_\beta]) = -\beta_a k(x_\alpha, x_\beta) \ . \tag{5.2.7}$$

Hence we have $(\alpha + \beta)_a k(x_\alpha, x_\beta) = 0$; since $\alpha + \beta \neq 0$, we obtain the result. (This result is similar to the part of the spectral theorem which says that eigenvectors relative to different eigenvalues are orthogonal.) □

It also follows that:

Lemma 5.11. *If* α *is a root,* $-\alpha$ *must be a root too.*

Proof. If this were not the case, an eigenvector x_α would be orthogonal to all of $\mathfrak{g}$; but this is impossible, because the Killing form is non-degenerate (theorem 5.6). □

As a particular case of (5.2.5), we now also see that

$$[L_\alpha, L_{-\alpha}] \subset \mathfrak{h} \ . \tag{5.2.8}$$

We also notice that the Killing form restricted to $\mathfrak{h}$,

$$k_{ab} \equiv k(H_a, H_b) \ , \tag{5.2.9}$$

is non-degenerate. Indeed, if there existed a $H \in \mathfrak{h}$ orthogonal to all other elements of $\mathfrak{h}$, it would be orthogonal to all of $\mathfrak{g}$ because of lemma 5.10. We will define k^{ab} to be the inverse of k_{ab}, so that

$$k^{ac} k_{cb} = \delta^a{}_b \ . \tag{5.2.10}$$

Let us also introduce an inner product of roots:

$$\alpha \cdot \beta \equiv \alpha_a k^{ab} \beta_b \ . \tag{5.2.11}$$

Example 5.3. *We have seen in example 5.2 that a Cartan subalgebra $\mathfrak{h}$ for $sl(N, \mathbb{C}) = \mathrm{su}(N)_\mathbb{C}$ is the subspace of diagonal traceless matrices. Let us consider $N = 3$: then the rank is $N - 1 = 2$. A possible basis for $\mathfrak{h}$ is given by*

$$H_1 \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad H_2 \equiv \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ . \tag{5.2.12}$$

(The overall coefficients have been chosen for convenience in the formulas below.) We now look for a basis of "creators" and "annihilators", similar to X and Y in (4.8.2) for $sl(2, \mathbb{C})$. A natural choice is simply

$$X_1 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad X_2 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad X_3 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ ; \qquad Y_i \equiv X_i^t \ . \tag{5.2.13}$$

This basis is indeed nice: it is easy to check that (5.2.1) is satisfied, with roots

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \qquad \alpha_2 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \ , \qquad \alpha_3 = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \ . \tag{5.2.14}$$

The roots for the Y_i are $-\alpha_i$; this confirms lemma 5.11. We can now compute k_{ab} as in (5.2.9). This is easy, because the action of ad_{H_a} is diagonal on the basis $\{H_a, X_i, Y_i\}$. We get:

$$k_{ab} = 2 \sum_{i=1}^{3} \alpha_{ia} \alpha_{ib} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \ . \tag{5.2.15}$$

*With this quadratic form, all roots α have the same length $\alpha \cdot \alpha = 1/3$. Moreover, the angle between α_1 and α_2 is (**exercise**: check this)*

$$\cos(\theta_{12}) = \frac{\alpha_1 \cdot \alpha_2}{|\alpha_1||\alpha_2|} = \frac{1}{2} \ , \tag{5.2.16}$$

where $|\alpha| \equiv \sqrt{\alpha \cdot \alpha} \equiv \sqrt{\alpha^2}$. We also have $\cos(\theta_{23}) = \frac{1}{2}$, $\cos(\theta_{13}) = -\frac{1}{2}$. So we can draw the α_I as three vectors in $\mathbb{R}^2$, of equal length, and with $\pi/3$ angles between them. We should also add the other roots $-\alpha_I$ to the drawing. We obtain the diagram in figure 9.

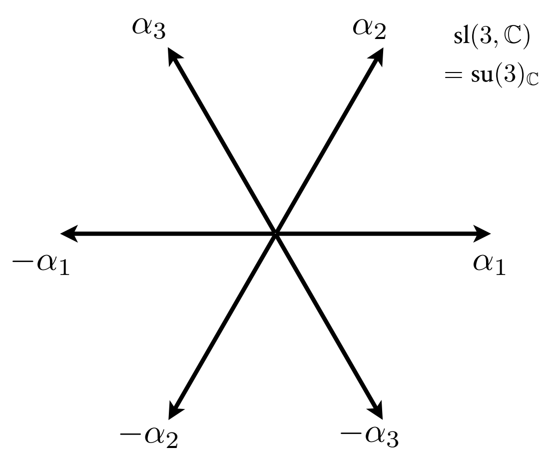


Figure 9: Roots for $\mathrm{sl}(3,\mathbb{C}) = \mathrm{su}(3)_{\mathbb{C}}$.

Thanks to the properties derived so far, we can already see that for each root α we can define an $\mathrm{sl}(2,\mathbb{C})$ subalgebra of $\mathfrak{g}$. The idea is that an element of L_α corresponds to the "creator" X in (4.8.3), an element of $L_{-\alpha}$ corresponds to the "annihilator" Y, and that a certain element of $\mathfrak{h}$ corresponds to H. To make this more precise, given $x_\alpha \in L_\alpha$, choose an element $x_{-\alpha} \in L_{-\alpha}$, normalized so that $k(x_{-\alpha}, x_\alpha) = 1$. Since

$$[x_\alpha, x_{-\alpha}] \in L_{\alpha-\alpha} = L_{\underline{0}} = \mathfrak{h} \ , \tag{5.2.17}$$

we see that $[x_\alpha, x_{-\alpha}] = a^a H_a$ for some coefficients a^a. To determine them, we consider

$$k_{ab}a^b = k(H_a, [x_\alpha, x_{-\alpha}]) = -k([x_\alpha, H_a], x_{-\alpha}) = \alpha_a k(x_\alpha, x_{-\alpha}) = \alpha_a \ . \tag{5.2.18}$$

So we have $a^a = k^{ab}\alpha_b$. To summarize,

$$[x_\alpha, x_{-\alpha}] = k^{ab}\alpha_a H_b \equiv \alpha \cdot H \ . \tag{5.2.19}$$

We can finally write the generators of a subalgebra $s_\alpha \subset \mathfrak{g}$ isomorphic to $\mathrm{sl}(2, \mathbb{C})$:

$$s_\alpha = \langle \{H_\alpha, X_\alpha, Y_\alpha\} \rangle \ ; \qquad H_\alpha = \frac{1}{\alpha^2} \alpha \cdot H \ , \qquad X_\alpha = \frac{1}{|\alpha|} x_\alpha \ , \qquad Y_\alpha = \frac{1}{|\alpha|} x_{-\alpha} \ , \tag{5.2.20}$$

where, again, $|\alpha| \equiv \sqrt{\alpha \cdot \alpha} \equiv \sqrt{\alpha^2}$. It is easy to verify that H_α, X_α and Y_α satisfy the commutation rules of H, X and Y in (4.8.3).

These subalgebras (one for each root α) will now help us derive a few more results about the set of roots Φ of an algebra. The idea is that we can decompose $\mathfrak{g}$ as a sum of irreducible representations of each s_α in (5.2.20).

Example 5.4. *We computed the roots of* $\mathfrak{g} = \mathrm{sl}(3, \mathbb{C}) = \mathrm{su}(3)_\mathbb{C}$ *in example 5.3. Let us see how the roots of* $\mathfrak{g}$ *decompose under the adjoint action of* s_{α_2}*. We can use lemma 5.9 to our advantage: the action of* X_{α_2} *will take* $L_{-\alpha_3}$ *to* L_{α_1}*, and* $L_{-\alpha_1}$ *to* L_{α_3}*. The action of* Y_{α_2} *will do just the opposite. So these two pairs of eigenspaces make up two representations of spin* $1/2$ *(each of dimension two) for* $s_{\alpha_2} \cong \mathrm{sl}(2, \mathbb{C})$*. We are left with four generators, in* L_{α_2}*,* $L_{-\alpha_2}$*, and* $\mathfrak{h}$*. The three generators* H_{α_2}*,* X_{α_2}*,* Y_{α_2} *live in these spaces, and form a representation of spin 1 (dimension 3) for* $s_{\alpha_2} \cong \mathrm{sl}(2, \mathbb{C})$*: it is its adjoint representation. The remaining generator of* $\mathfrak{h}$ *forms a singlet representation (spin 0, dimension 1). Summing up, we have that the adjoint of* $\mathrm{su}(3)_\mathbb{C}$ *decomposes as* $8 = 3 \oplus 2 \oplus 2 \oplus 1$ *under the subalgebra* s_{α_2}*.*

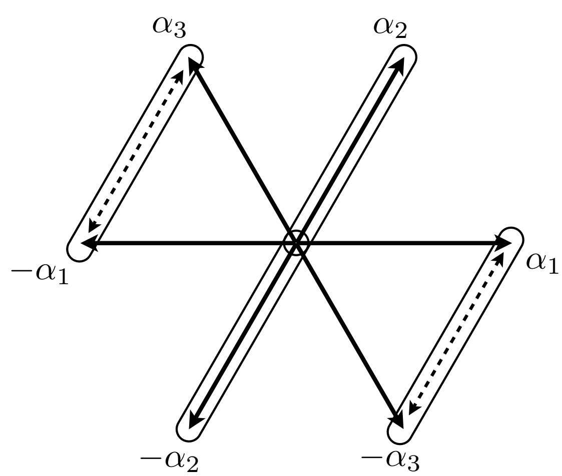


Figure 10: The decomposition of $\mathrm{sl}(3, \mathbb{C}) = \mathrm{su}(3)_\mathbb{C}$ under the subalgebra s_{α_2}.

Lemma 5.12. *If* $\alpha \neq 0$*,* L_α *has dimension 1. On the other hand,* $L_{k\alpha}$ *is the empty set for* $k \neq -1, 0, 1$*.*

Proof. Let us decompose $\mathfrak{h} \oplus (\oplus_k L_{k\alpha})$ in irreducible representations of the subalgebra s_α in (5.2.20), isomorphic to $\mathrm{sl}(2, \mathbb{C})$. First of all, the subspace of $\mathfrak{h}$ that commutes with all of s_α has dimension $r - 1$: all the elements of the form $\sum_a a_a H_a$ commute with x_α and $x_{-\alpha}$ if $\sum_a a_a \alpha_a = 0$. So there is a basis for $\mathfrak{h}$ in which one generator, H_α, does not commute with s_α, and all other generators commute with it.

Let us now suppose that L_α has more than one element: say it has two, x_α and x'_α. We know that there is at least an element $x_{-\alpha} \in L_{-\alpha}$ (from lemma 5.11). We see then that $\mathrm{ad}_{H_\alpha} x'_\alpha = x'_\alpha$. (This computation works in the same way as to the proof of $[H_\alpha, X_\alpha] = X_\alpha$.) This seems already suspicious: x_α, $x_{-\alpha}$ and H_α already span a spin one representation of s_α (the adjoint). If there existed a second element x'_α with $m = 1$, we wouldn't be able to complete it to a full representation of s_α, because we wouldn't have an $m = 0$ vector to add to it.

Here is another argument. By taking linear combinations, we can suppose that $k(x'_\alpha, x_{-\alpha})$=0. Now, we know that $[x'_\alpha, x_{-\alpha}]$ is in $\mathfrak{h}$, which means that it is a linear combination of the H_a. In order to find the coefficients, we can reason in a similar way to (5.2.18):

$$k(H_a, [x'_\alpha, x_{-\alpha}]) = -k([x'_\alpha, H_a], x_{-\alpha}) = \alpha_a k(x'_\alpha, x_{-\alpha}) = 0 \ . \tag{5.2.21}$$

It follows that the coefficients vanish, and hence that

$$[x'_\alpha, x_{-\alpha}] = 0 \ . \tag{5.2.22}$$

So x'_α is annihilated by ad_{Y_α} in (5.2.20), which is proportional to $\mathrm{ad}_{x_{-\alpha}}$. But then x'_α would have minimal weight m. But we also know already that it has $m = 1$. This is a contradiction: the minimal weight m in a representation of $\mathrm{sl}(2, \mathbb{C})$ should be either negative, or zero (see (4.8.14)).

To see that $L_{k\alpha}$ is empty for $k \neq -1, 0, 1$, it is now enough to remark that $x_{k\alpha}$ would have eigenvalue k under H_α; so its $m = k$, which means that it would belong to a representation of spin $l > 1$. However, x_α is already a highest weight state, since $\mathrm{ad}_{X_\alpha} x_\alpha = 0$. So the only way to have a representation of spin $l > 1$ would be to add more states of $m = 1, 0, -1$. But we have already shown that such states cannot exist: the only state of $m = 1$ for s_α is x_α, the only state with $m = 0$ is H_α, the only state with $m = -1$ is $x_{-\alpha}$. □

We can generalize this further:

Theorem 5.13. *The eigenvalue m_α under the $sl(2,\mathbb{C})$ algebra s_α of a general $x_\beta \in \mathfrak{g}$ is given by:*

$$m_\alpha(x_\beta) = \frac{\alpha \cdot \beta}{\alpha^2} \ . \tag{5.2.23}$$

Proof.

$$\mathrm{ad}_{H_\alpha} x_\beta = [H_\alpha, x_\beta] = \frac{\alpha \cdot \beta}{\alpha^2} x_\beta \ . \tag{5.2.24}$$

□

Since the eigenvalue m of ad_H for a finite-dimensional representation of $\mathrm{sl}(2,\mathbb{C})$ must be in $\frac{1}{2}\mathbb{Z}$ (see again (4.8.14)), we obtain

Lemma 5.14. *If α and β are two roots,*

$$2\frac{\alpha \cdot \beta}{\alpha^2} \in \mathbb{Z} \ . \tag{5.2.25}$$

This is also true if we exchange the roles of α and β. Thus we see that

$$\cos^2(\theta_{\alpha,\beta}) = \frac{(\alpha \cdot \beta)^2}{\alpha^2\, \beta^2} \in \frac{\mathbb{Z}}{4} \ . \tag{5.2.26}$$

This gives strong constraints on the possible angles between two roots, as the following example illustrates.

Example 5.5. *Figure 11 gives us a graphical representation of the roots for four semisimple Lie algebras of rank 2. These diagrams are drawn in a basis in which the quadratic form k_{ab} is δ_{ab}. We have already discussed* $\mathrm{sl}(3,\mathbb{C})$ *in example 5.3. The two diagrams on the left correspond to algebras we know already,* $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$ *and* $\mathrm{so}(5,\mathbb{C})$*. We won't see the computations here, but we can check that the dimensions are correct: according to (5.2.3), the dimension should be the number of non-zero roots plus* $\dim(\mathfrak{h}) = r$*. This gives* $4 + 2 = 6$ *for* $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$ *and* $8 + 2 = 10$ *for* $\mathrm{so}(5,\mathbb{C})$*, which are correct. Finally, the algebra g_2 is a* $12 + 2 = 14$*-dimensional Lie algebra which is new to us. We will see together that these four are* the only *Lie algebras with these properties.*

We see that lemma 5.12 is satisfied for all these algebras: no multiple of a root α is a root, other than $-\alpha$. It is more fun to see what theorem 5.13 tells us; as a visual aid, we have already labelled two roots α and β.

- *For* $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$*, we get* $\mathrm{ad}_{H_\alpha}(x_\beta) = \frac{\alpha \cdot \beta}{|\alpha^2|} x_\beta = 0$*. So $m_\alpha = 0$ for the $sl(2,\mathbb{C})$ subalgebra s_α. In fact, we see that there is no other root of the form $\beta + k\alpha$, so β is in a singlet (“spin 0”) representation under s_α.*

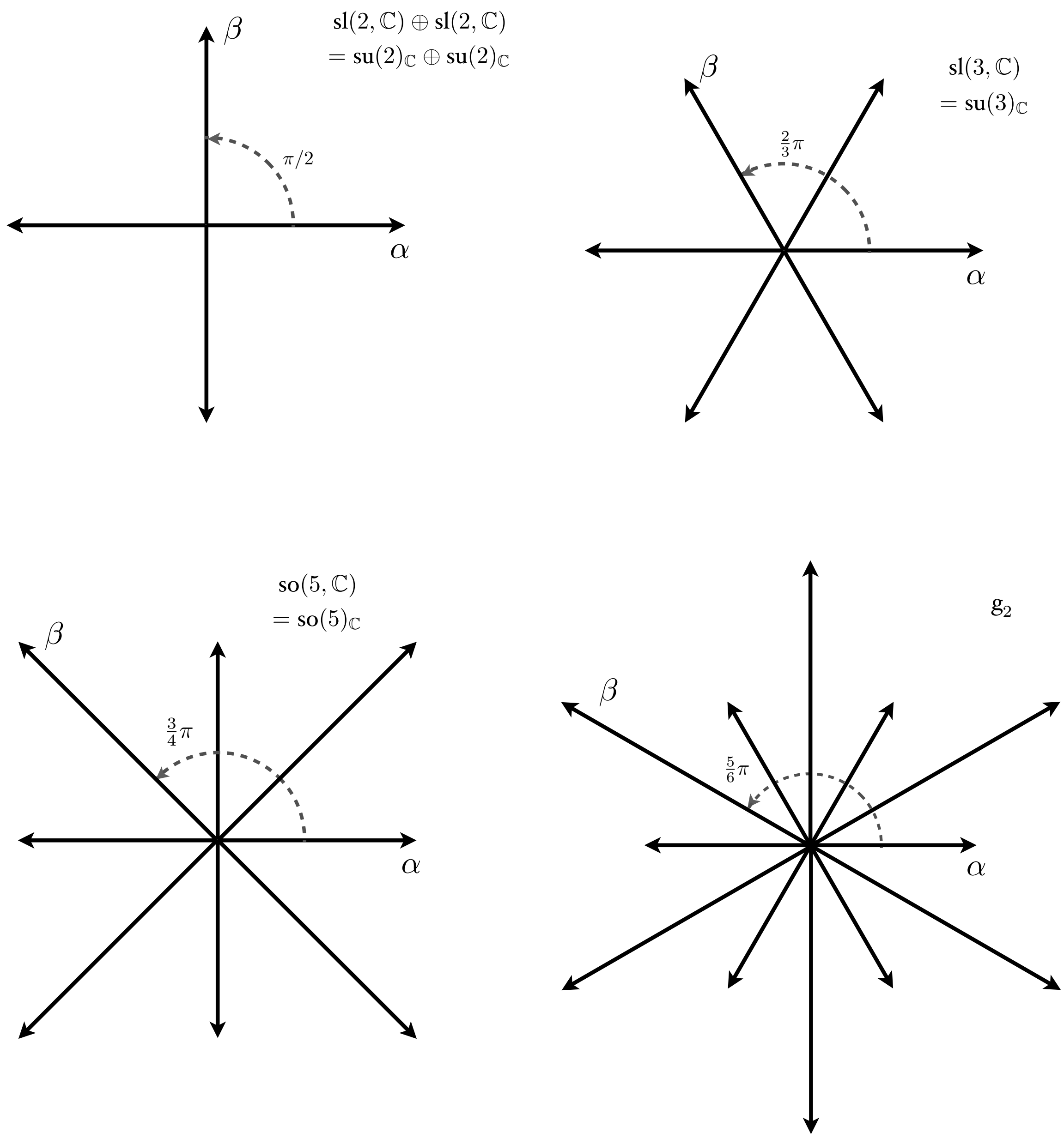


Figure 11: Roots of all rank 2 semisimple Lie algebras.

- *For* $\mathrm{sl}(3,\mathbb{C})$*, we get* $\mathrm{ad}_{H_\alpha}(x_\beta) = \frac{\alpha\cdot\beta}{|\alpha^2|}x_\beta = -\frac{1}{2}$*, as we already computed in example* 5.3*. So* x_β *has* $m = -\frac{1}{2}$*; there has to be at least another root with* $m = \frac{1}{2}$*. Indeed there is one: it is* $\alpha+\beta$*. Together,* β *and* $\alpha+\beta$ *make up a representation of spin* $\frac{1}{2}$ *under* σ_α*.*

- *For* $\mathrm{so}(5,\mathbb{C})$*, this computation is a bit more subtle because* α *and* β *have different lengths. If we normalize* α *to have length one (the overall length of the roots is not important in this computation), the quickest way to compute this is to see that* $\beta+\alpha$ *is orthogonal to* α*, so that it has* $m=0$*; then* β *has* $m=-1$*, and* $\beta+2\alpha$ *has* $m=1$*. Together,* β*,* $\beta+\alpha$ *and* $\beta+2\alpha$ *make up a representation of* s_α *of spin 1.*

- *Finally, for* g_2*, we can compute from the figure that* x_β *has* $m_\alpha = -3/2$*; the four roots* $\beta+k\alpha$*,* $k=0,1,2,3$*, make up a representation of* s_α *of spin 2.*

In example 5.5, we assumed the quadratic form k_{ab} could be brought to the form δ_{ab}. The following theorem tells us that this is always possible:

Theorem 5.15. *The vector space generated by real linear combinations of the roots has dimension* r*:* $\mathbb{R}^r = \mathrm{Span}_{\mathbb{R}}(\Phi)$*. In other words, there exists a basis in* $\mathfrak{h}$ *such that all the roots have real coefficients.*[20] *The quadratic form* k_{ab} *is positive definite on this space.*

Proof. We want to compute the Killing form (4.8.24), $k_{ab} = k(H_a,H_b) = \mathrm{Tr}(\mathrm{ad}_{H_a}\mathrm{ad}_{H_b})$, in a convenient basis: the one given by (5.2.3). ad_{H_a} gives zero on $L_0 = \mathfrak{h}$, while it gives α_i on each L_α. Recalling that each L_α has dimension 1 (lemma 5.12), we get

$$k_{ab} = k(H_a,H_b) = \mathrm{Tr}(\mathrm{ad}_{H_a}\mathrm{ad}_{H_b}) = \sum_{\alpha\in\Phi}\alpha_a\alpha_b \ . \tag{5.2.27}$$

(Notice that we have already used this method in example 5.3.) We now multiply this equation by $\beta^a\beta^b$, where $\beta^a \equiv k^{ab}\beta_b$ and β is a root. We find

$$\beta\cdot\beta = \sum_{\alpha\in\Phi}(\alpha\cdot\beta)^2 \ . \tag{5.2.28}$$

We now divide by $(\beta\cdot\beta)^2$:

$$\frac{1}{\beta\cdot\beta} = \sum_{\alpha\in\Phi}\frac{(\alpha\cdot\beta)^2}{(\beta\cdot\beta)^2} \ . \tag{5.2.29}$$

[20] For example, the space $\mathrm{Span}_{\mathbb{C}}$ of complex linear combinations of the vectors $(1,1)$, $(1,i)$, $(1,-1)$ is $\mathbb{C}^2$, while the space $\mathrm{Span}_{\mathbb{R}}$ of their *real* linear combinations is $\mathbb{R}^3$. On the other hand, $\mathrm{Span}_{\mathbb{C}}((1,1),(1,0),(1,-1)) = \mathbb{C}^2$, and $\mathrm{Span}_{\mathbb{R}}((1,1),(1,0),(1,-1)) = \mathbb{R}^2$.

Thanks to (5.2.25), we know that each of the summands on the right hand side is rational. It then follows that $\beta \cdot \beta$ is rational. But then (5.2.25) also tells us that $\alpha \cdot \beta$ is rational, for each roots α and β. In particular, $\alpha \cdot \beta$ is real. This means that the projection of any root on any other is real, which implies that there is a basis in which all roots are real. This proves the first statement in the theorem.

Since we now know that $\alpha \cdot \beta$ is real, (5.2.28) implies that $\beta \cdot \beta$ is a sum of squares of real numbers. We then conclude

$$\beta \cdot \beta > 0 \tag{5.2.30}$$

for any root β. □

We conclude this list of properties with:

Lemma 5.16. *If α and β are roots, the reflection*[21] *of β with respect to α:*

$$w_\alpha(\beta) = \beta - 2\frac{\beta \cdot \alpha}{\alpha^2}\alpha \tag{5.2.31}$$

is a root.

Proof. Let us again decompose $\mathfrak{g}$ in irreducible representations of the subalgebra s_α defined in (5.2.20). In particular, we have seen that the weight m_α of x_β under this subalgebra is given by (5.2.23). This means x_β belongs to a representation with $l \geq |m_\alpha|$. This representation will also contain a root with weight $-m_\alpha$ under s_α. But this is exactly the weight of $w_\alpha(\beta)$:

$$\frac{\alpha \cdot w_\alpha(\beta)}{\alpha^2} = \frac{\alpha \cdot \beta}{\alpha^2} - 2\frac{\alpha \cdot \beta}{(\alpha^2)^2}\alpha^2 = -\frac{\alpha \cdot \beta}{\alpha^2} = -m_\alpha \, . \tag{5.2.32}$$

□

Definition 5.6. *The finite group generated by reflections of the form w_α in (5.2.31), for $\alpha \in \Phi$, is called the* Weyl group.

Let us now summarize what we have seen in this subsection. The roots of a Lie algebra of rank r generate a space $\mathbb{C}^r$ (see (5.2.4)). If we take their real combinations, we obtain a subspace $\mathbb{R}^r$. On this subspace, k_{ab} is a positive definite quadratic form (lemma 5.15). The set of all non-zero roots enjoys the properties:

[21] It is a "reflection" in the sense that it reflects $\mathfrak{h}$ with respect to the plane orthogonal to α. To see this, it is enough to notice that, if β belongs to that plane, $\beta \cdot \alpha = 0$, hence $\beta' = \beta$; and that, on the other hand, $\alpha' = -\alpha$.

- If α is a root, $-\alpha$ is also a root, but no other multiple is (see lemma 5.11, and second part of lemma 5.12).
- If α, β are roots, $2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha} \in \mathbb{Z}$ (lemma 5.14).
- The set of all roots is left invariant by the Weyl group generated by the reflections w_α in (5.2.31).

To any non-zero root α is associated a $L_\alpha \subset g$ of dimension 1 (lemma 5.12).

Finally, notice that we know by now how to reconstruct a Lie algebra from the set of its roots: it is enough to use (5.2.5) and (5.2.19).

5.3 Simple roots

We will now use the properties of the roots to classify all semisimple Lie algebras. We will do so by introducing a basis for all the roots (over $\mathbb{Z}$: namely, we will write all roots as a linear combination of elements of the basis with *integer* coefficients).

A convenient basis is given by "simple" roots. The idea is to first introduce a concept of "positive" roots, by dividing the space $\mathbb{R}^r$ in two by a plane, and then to pick the roots which are "closest" to this plane.

The choice of the plane is not important:

Definition 5.7. *We say that a root α is* positive *if its first coordinate $\alpha_1 > 0$; or, in case $\alpha_1 = \ldots = \alpha_k = 0$, if the first non-zero coordinate $\alpha_{k+1} > 0$. A root α is* negative *if $-\alpha$ is positive. We will denote by Φ_+ the set of all positive roots, and by Φ_- the set of all negative roots.*

This definition of positivity gives also rise to an ordering: we say that $\alpha > \beta$ if $\alpha - \beta$ is positive.

Definition 5.8. *A positive root α is said to be* simple *if it cannot be written as a sum of two positive roots. We will denote by Π the set of all simple roots.*

Example 5.6. *The roots we called α and β in figure 11 are possible choices of simple roots.*

Lemma 5.17. *If α and β are simple roots, $\alpha - \beta$ is not a root.*

Proof. $\alpha-\beta > 0$ is impossible, because otherwise α would be sum of the two positive roots β and $\alpha - \beta$ — which would be in contradiction with the assumption that α is simple.

$\alpha - \beta < 0$ is also impossible, because otherwise β would be sum of the two positive roots α and $\beta - \alpha$. □

Lemma 5.18. *The inner product of two simple roots α and β is non-positive:*

$$\alpha \cdot \beta \leq 0 \ . \tag{5.3.1}$$

Proof. We know that $[x_{-\alpha}, x_\beta] = 0$ (lemmas 5.17 and 5.9). Since $x_{-\alpha}$ is proportional to the "annihilator" Y_α in s_α (see (5.2.20)), it follows that x_β is the lowest weight state for s_α: so $m_\alpha \leq 0$. But then, from (5.2.23) and theorem 5.15, we have (5.3.1). □

Theorem 5.19. *Any positive root can be written as a linear combination of simple roots with non-negative coefficients: $\alpha = \sum_{a=1}^r x_a \alpha_a$, $x_a \geq 0$. Given a choice $\{\alpha_a\}$ of simple roots, the coefficients x_a are unique.*

Proof. The first statement is certainly true for simple roots. A root which is positive but not simple can be written as a sum of two positive roots (or else it would be simple). Continuing this process by induction, we can find the claimed linear combination. Since at each step we have a sum, and not a difference, the coefficients are positive.

We now show that the coefficients are unique. It is enough to show that simple roots are *linearly independent*. If there existed a linear combination $\sum_a x_a \alpha_a = 0$, some coefficients x_a would be positive (call them x_a^+), and some other negative (write them as $-x_a^-$, where $x_a^- > 0$). We could then write

$$\sum_a x_a^+ \alpha_{a+} = \sum_b x_b^- \alpha_{b-} \equiv \beta \ . \tag{5.3.2}$$

It would then follow that $\beta^2 = (\sum x_a^+ \alpha_{a+}) \cdot (\sum x_b^- \alpha_{b-}) = \sum_{a,b} x_a^+ x_b^- \alpha_{a+} \cdot \alpha_{b-}$. But both x_a^+ and x_b^- are positive; moreover, all of the products $\alpha_{a+} \cdot \alpha_{b-} < 0$, because of (5.3.1). We would then conclude that $\beta^2 < 0$, which would be in contradiction with theorem 5.15. □

Given a choice $\{\alpha_a\}$ of simple roots, let us now introduce the *Cartan matrix*

$$C_{ab} \equiv 2 \frac{\alpha_a \cdot \alpha_b}{\alpha_b \cdot \alpha_b} \ , \tag{5.3.3}$$

whose entries are integer, thanks to (5.2.25). Notice that this matrix is not necessarily symmetric (because of its denominator).

Theorem 5.20. *A semisimple Lie algebra can be completely reconstructed from its Cartan matrix.*

Proof. Thanks to theorem 5.19, we know that C_{ab} is an $r \times r$ matrix (where r is the rank). The simple roots will then make up a basis in $\mathbb{R}^r$. The angles between any two roots will be given by

$$\cos^2(\theta_{\alpha_a,\alpha_b}) = \frac{(\alpha_a \cdot \alpha_b)^2}{\alpha_a^2\, \alpha_b^2} = \frac{1}{4} C_{ab} C_{ba} \; ; \tag{5.3.4}$$

the ratio between the lengths squared of two α_a can instead be obtained from

$$\frac{\alpha_a^2}{\alpha_b^2} = \frac{C_{ab}}{C_{ba}} \; . \tag{5.3.5}$$

From this information, we know how to draw all the simple roots.

We should now reconstruct all the other roots. First of all, recall that the α_a are annihilated by the action ad_α of any other positive root α. Hence α_a is a lowest-weight state for the $\mathrm{sl}(2,\mathbb{C})$ subalgebra s_α. Moreover, the weight of α_a under s_α is given by $-l = \frac{\alpha \cdot \alpha_a}{\alpha^2}$ (see (5.2.23)). Hence, we should add roots of the form $\alpha_i + k\alpha$, $k = 0, \ldots, 2l$, to complete α_a to a representation for s_α.

Iterating this process, we can obtain all positive roots α; we can then add all the negative roots $-\alpha$. Finally, from the set of all roots we can reconstruct the whole Lie algebra, as observed at the end of section 5.2. □

Example 5.7. *Suppose we are given a Cartan matrix*

$$C_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \; . \tag{5.3.6}$$

Since this is a 2×2 matrix, we see that the rank is 2. From (5.3.5) we see that the two simple roots (call them α and β) have equal length. From (5.3.4) we see that the angle θ between these two satisfies $\cos^2(\theta) = \frac{1}{4}$. Since the inner product of two simple roots is non-positive (lemma 5.18), we have $\cos(\theta) = -\frac{1}{2}$. So α and β are two vectors of equal length and at an angle of $2\pi/3$. We now compute the weight m_α of the state x_β: it is given by (5.2.23), which is exactly $\frac{1}{2}C_{12} = -\frac{1}{2}$. So x_β belongs to the representation of spin $1/2$ under s_α. Hence we have to add the root $\alpha + \beta$ to complete the representation. It is easy to see that there are no more positive roots to add. If we now add the negative roots $-\alpha$, $-\beta$, $-\alpha - \beta$, we end up with the diagram in the upper-right corner of figure 11 — which corresponds to $sl(3,\mathbb{C})$.

5.4 Dynkin diagrams

We have just seen (theorem 5.20) that the structure of a semisimple Lie algebra can be encoded in the Cartan matrix (5.3.3). We can actually condense this information further in a single diagram called *Dynkin diagram*.

Here is the procedure:

- Draw a point (or "node") for each simple root (hence, a total of r points).
- Connect node a to node b by $C_{ab}C_{ba}$ lines. (We know from (5.3.4) and (5.2.26) that the number of lines can only be $0, 1, 2, 3$.)
- If the number of lines between two nodes is 2 or 3, it follows that C_{ab} is 2 or 3, and $C_{ba} = 1$ (or viceversa). Then (5.3.5) tells us that one of the roots is longer than the other. Overlap a ">" sign to the lines connecting these two nodes, showing which of the nodes corresponds to the longer root.

(The graph obtained with the first two rules alone, without the ">" sign, is sometimes called *Coxeter graph*.)

Example 5.8. *From these rules, we see immediately that there are only four possible rank 2 examples: two nodes connected by* $0, 1, 2$ *or* 3 *lines. These are respectively* $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$*,* $\mathrm{sl}(3,\mathbb{C})$*,* $\mathrm{so}(5,\mathbb{C})$*,* g_2*, which are the four examples shown in figure 11.*

Things get more complicated for higher rank. Not any diagram obtained by connecting dots with lines is the Dynkin diagram of a semisimple Lie algebra. For example, one can show that a Dynkin diagram has no loops. Using this and other properties, one can actually give a complete classification. In figure 12 we give the complete list for *simple* Lie algebras. Each diagram has a "traditional" label, which is also shown. Since semisimple Lie algebras are simply direct sums of simple ones (lemma 5.2), this list of diagrams gives a complete classification.

Notice that $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$, which corresponds to two nodes without any lines, is not included in figure 12 because it is semisimple but not simple: both $\mathrm{sl}(2,\mathbb{C})$ summands are non-trivial invariant subalgebras, although they are not abelian invariant subalgebras. The other three rank 2 examples considered in figure 11 and in example 5.5 are all included in figure 12 as particular cases, as we will see shortly.

Let us first go over the list of algebras systematically. There are three types of diagrams in figure 12 that encode Lie algebras we know already:

- A_n corresponds to the algebra $\mathrm{sl}(n+1,\mathbb{C}) = \mathrm{su}(n+1)_{\mathbb{C}}$.

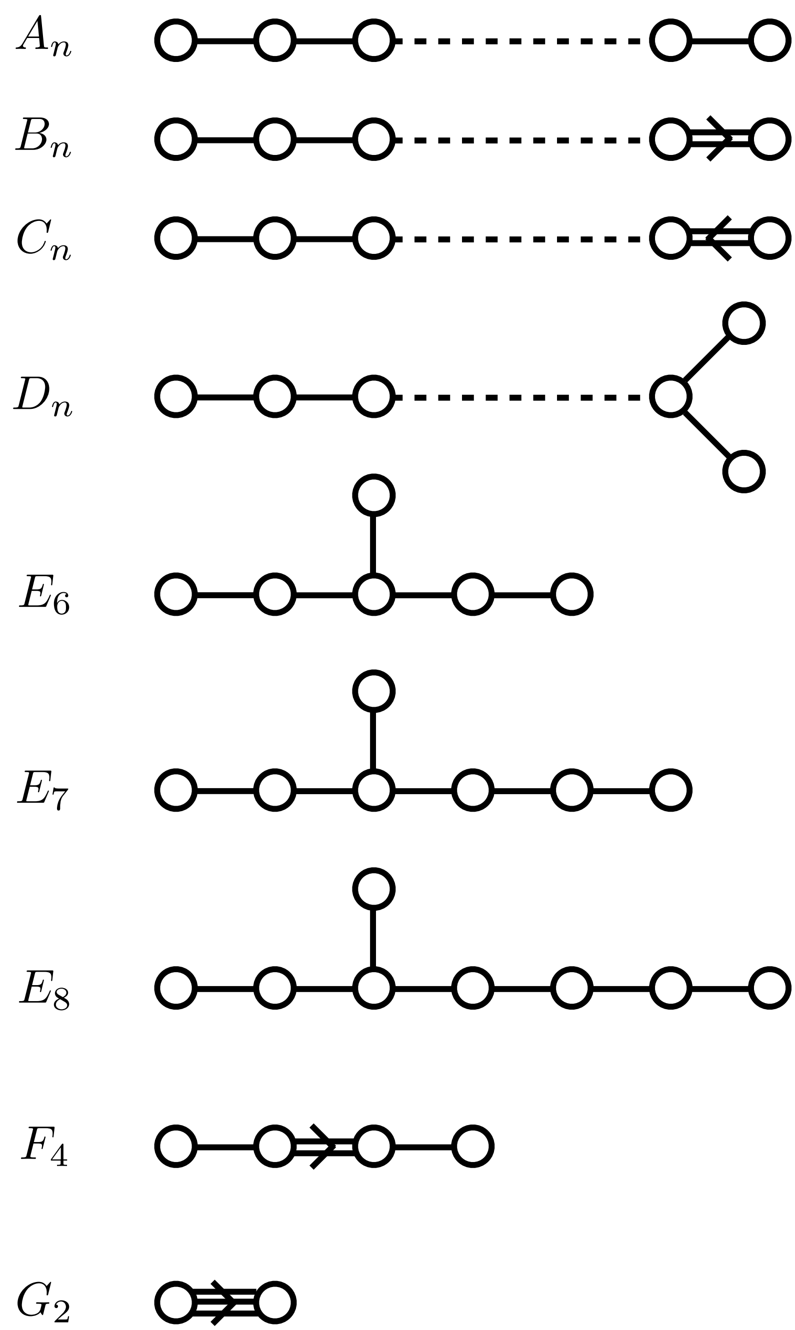


Figure 12: All Dynkin diagrams for *finite-dimensional* simple Lie algebras. The index $_n$ refers to the total number of nodes (which is equal to the rank).

- B_n corresponds to the algebra $\mathrm{so}(2n+1,\mathbb{C}) = \mathrm{so}(2n+1)_{\mathbb{C}}$. Notice that diagram B_1 coincides with diagram A_1. This is because $\mathrm{so}(3,\mathbb{C}) \cong \mathrm{sl}(2,\mathbb{C})$, which is the complexification of the familiar statement that $\mathrm{so}(3) \cong \mathrm{su}(2)$.

- D_n corresponds to the algebra $\mathrm{so}(2n,\mathbb{C}) = \mathrm{so}(2n)_{\mathbb{C}}$. In this case, we require $n \geq 3$, since neither $\mathrm{so}(2)$ nor $\mathrm{so}(4)$ are simple.

  Amusingly, however, the diagram D_2 would have two nodes unconnected by any lines, which is the same as the diagram for $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$; indeed it happens that $\mathrm{so}(4,\mathbb{C}) \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$ (compare with lemma 4.9).

  Notice also that the diagram D_3 is identical to A_3. Indeed we have $\mathrm{so}(6,\mathbb{C}) \cong \mathrm{sl}(4,\mathbb{C})$. At the level of real Lie algebras, $\mathrm{so}(6) \cong \mathrm{su}(4)$. In fact, there is a two-to-one homomorphism of $\mathrm{SU}(4)$ into $\mathrm{SO}(6)$; one can think of $\mathrm{SU}(4)$ as of a "spin group" $\mathrm{Spin}(6)$ in six Euclidean dimensions, defined similarly to (4.10.55). So we have $\mathrm{SO}(6) \cong \mathrm{SU}(4)/\mathbb{Z}_2$.

The diagram C_n corresponds to a series of Lie algebras we haven't considered so far, $\mathrm{sp}(n,\mathbb{C})$:

Definition 5.9. *A $2n \times 2n$ matrix S is called* symplectic *if it preserves the antisymmetric form J:*

$$S^t J S = J \ , \qquad J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \ . \tag{5.4.1}$$

We can easily check that the set of symplectic matrices over a field $\mathbb{F}$ (we will consider either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) is a group:

$$\mathrm{Sp}(n,\mathbb{F}) \equiv \left\{ S \in \mathrm{Mat}(2n,\mathbb{F}) \mid S^t J S = J \right\} \ . \tag{5.4.2}$$

Expanding $S \sim 1_{2n} + s$ as in section 4.3, we see that the corresponding Lie algebra is

$$\mathrm{sp}(n,\mathbb{F}) \equiv \left\{ s \in \mathrm{Mat}(2n,\mathbb{F}) \mid s^t J + J s = 0 \right\} \ . \tag{5.4.3}$$

Clearly, $\mathrm{sp}(n,\mathbb{R})_{\mathbb{C}} = \mathrm{sp}(n,\mathbb{C})$. But we will see now another real Lie algebra with the same complexification.

Definition 5.10. *Recall the* quaternions*:*

$$\mathbb{H} = \left\{ a_0 + i a_1 + j a_2 + k a_3, \ a_i \in \mathbb{R} \right\} \ , \qquad i^2 = j^2 = k^2 = ijk = -1 \ . \tag{5.4.4}$$

The conjugate of a quaternion $h = a_0 + ia_1 + ja_2 + ka_3$ *is* $\bar{h} \equiv a_0 - ia_1 - ja_2 - ka_3$*. For a matrix with quaternion entries, we have* $M^\dagger_{ij} = \overline{M}_{ji}$*, just like for complex numbers. We then define*

$$\mathrm{Sp}(n) \equiv \left\{ S \in \mathrm{Mat}(n, \mathbb{H}) \mid S^\dagger S = 1_n \right\} . \tag{5.4.5}$$

It turns out that also $\mathrm{sp}(n)_{\mathbb{C}} = \mathrm{sp}(n, \mathbb{C})$.

For $n = 1$, the diagram C_1 is the same as A_1 (and B_1). Indeed we have $\mathrm{sp}(1, \mathbb{C}) \cong \mathrm{sl}(2, \mathbb{C})$. At the level of *real* Lie algebras, $\mathrm{sp}(1) \cong \mathrm{su}(2)$. In fact, we even have

$$\mathrm{Sp}(1) \cong \mathrm{SU}(2) . \tag{5.4.6}$$

For $n = 2$, the diagram C_2 coincides with B_2. Indeed we have $\mathrm{sp}(2, \mathbb{C}) \cong \mathrm{so}(5, \mathbb{C})$. At the level of real Lie algebras, $\mathrm{sp}(2) \cong \mathrm{so}(5)$. In fact, there is a two-to-one homomorphism of $\mathrm{Sp}(2)$ into $\mathrm{SO}(5)$; one can think of $\mathrm{Sp}(2)$ as of a "spin group" $\mathrm{Spin}(5)$ in five Euclidean dimensions, defined similarly to (4.10.55). So we have $\mathrm{SO}(5) \cong \mathrm{Sp}(2)/\mathbb{Z}_2$.

This leaves us with the five diagrams $E_{6,7,8}$, F_4, G_2, which correspond to five Lie algebras we have not seen before. Since they are not part of any infinite series, they are called *exceptional*. The diagram G_2 corresponds to the rank 2 root system in the lower-right corner of figure 11.

5.5 Representations of semisimple Lie algebras

We will now use the formalism of section 5 to classify *representations* of semisimple Lie algebras. For example, we would like to know whether the representations of $\mathrm{SU}(N)$ we saw in section 4.9, which we have studied in terms of tensors, are the only possible representations.

The following theorem is a variation on theorem 4.3.

Theorem 5.21. ***(Weyl)***. *Every decomposable representation of a finite-dimensional semisimple Lie algebra is reducibile.*

Because of this, once again we will focus on *irreducible* representations.

Our classification will be very similar to our discussion of roots in section 5.2. Recall that a crucial concept was the Cartan subalgebra $\mathfrak{h}$: it is an abelian subalgebra, whose elements H_a are diagonalizable in the adjoint representation. Using theorem 5.21, one can show that the H_a are diagonalizable in every representation; in other words, $\rho(H_a)$ is a diagonalizable matrix.

Since the H_a commute, we have $0 = \rho([H_a, H_b]) = [\rho(H_a), \rho(H_b)]$. Hence the matrices $\rho(H_a)$ are simultaneously diagonalizable. We can then split the vector space V on which the representation ρ acts in eigenspaces V_μ of the Cartan subalgebra $\mathfrak{h}$. This generalizes the procedure in section 5.2: in that case, ρ was the adjoint representation ad, and V was $\mathfrak{g}$ itself.

For an eigenvector v_μ, we can write

$$\rho(H_a)v_\mu = \mu_a v_\mu \ . \tag{5.5.1}$$

Recall that the index a runs from 1 to the rank r. The μ_a can be thought of as the components of a vector $\mu \in \mathbb{C}^r$, that we are going to call **weight**. We can then decompose our representation as a direct sum of eigenspaces:

$$V = \oplus_\mu V_\mu \ . \tag{5.5.2}$$

Example 5.9. *We studied in example 4.21 the representations of* $\mathrm{sl}(2,\mathbb{C})$*. We labeled each vector* $|m\rangle$ *in the representation by a half-integer* m*, such that* $H|m\rangle = m|m\rangle$*. Comparing with (5.5.1), we see that this* m *is then an example of weight. It is a one-dimensional vector, because* $\mathrm{sl}(2,\mathbb{C})$ *has rank 1. A* V_μ *in this case is simply the span of one such state,* $\langle\ |m\rangle\ \rangle$*. The whole representation can be written as the direct sum of all these one-dimensional eigenspaces.*

Example 5.10. *For any semisimple Lie algebra, the weight of the adjoint representation are simply the roots.*

To make our next formulas more readable, from now on we will drop the symbol ρ for the representation, just as we did in example 4.21. Hence, when the context is clear, instead of $\rho(x)$ we will just write x:

$$\rho(x)v \rightsquigarrow x\,v \ . \tag{5.5.3}$$

Expressions such as $[x_1, x_2]v$ should not give rise to confusion: they can be understood both as $[\rho(x_1), \rho(x_2)]v$ (where the commutator is the matrix commutator) and as $\rho([x_1, x_2])v$ (where the commutator is the abstract Lie algebra commutator) — but, since ρ is a Lie algebra homomorphism, these two ways of reading $[x_1, x_2]v$ are equal.

Lemma 5.22. *If* α *is a root and* μ *is a weight,*

$$\rho(L_\alpha)V_\mu \subset V_{\alpha+\mu} \tag{5.5.4}$$

Proof.

$$H_a(x_\alpha v_\mu) - x_\alpha \mu_a v_\mu = [H_a, x_\alpha] v_\mu = \alpha_a x_\alpha v_\mu \tag{5.5.5}$$

and so

$$H_a(x_\alpha v_\mu) = (\alpha + \mu)_a x_\alpha v_\mu \ . \tag{5.5.6}$$

This means that $x_\alpha v_\mu$ has weight $\alpha+\mu$, which is what we had to show. Notice how similar this proof is to the one of lemma 5.9. □

As in lemma 5.9, if $\alpha+\mu$ is not a weight, (5.5.4) is still true, in the sense that $V_{\alpha+\mu} = \varnothing$.

Example 5.11. *For $sl(2,\mathbb{C})$, recall that the roots are 1 and -1. Lemma 5.22 simply says that $X|m\rangle = |m+1\rangle$, and $Y|m\rangle = |m-1\rangle$.*

Lemma 5.23. *If two representations ρ_1 and ρ_2 have weights $\{\mu^1_I\}$ and $\{\mu^2_J\}$, the representation $\rho_1 \otimes \rho_2$ has weights given by all the possible sums $\{\mu^1_I + \mu^2_J\}$.*

Proof. For the group G, recall that the tensor product representation $(\rho_1 \otimes \rho_2)(g) \equiv \rho_1(g) \otimes \rho_2(g)$. If we expand $g \sim 1 + x$, we see that, for the Lie algebra $\mathfrak{g}$, $(\rho_1 \otimes \rho_2)(x) = \rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)$. For the Cartan subalgebra, then,

$$(\rho_1 \otimes \rho_2)(H_a) v_{\mu_I} \otimes v_{\mu_J} = (\rho_1(H_a) \otimes 1 + 1 \otimes \rho_2(H_a)) v_{\mu_I} \otimes v_{\mu_J} = (\mu_{I,a} + \mu_{J,a}) v_{\mu_I} \otimes v_{\mu_J} \ . \tag{5.5.7}$$

□

Example 5.12. *We have seen in section 4.9 some representations of* su(3)*: by complexification, these determine representations of* sl$(3,\mathbb{C})$*.*

- *We start with the fundamental of* su(3). *The $\rho(H_a)$ are simply the H_a given in (5.2.12): they are already diagonal. So we can read off the weights without any computation. For example, the vector μ_1 is given by the eigenvalues with respect to H_1 and H_2 of the simultaneous eigenvector $(1,0,0)^t$; so $\mu_1 = \left(\begin{smallmatrix} 1/2 \\ 1/2\sqrt{3} \end{smallmatrix}\right)$. Summing up:*

$$\mu_1 = \begin{pmatrix} 1/2 \\ 1/2\sqrt{3} \end{pmatrix} , \qquad \mu_2 = \begin{pmatrix} -1/2 \\ 1/2\sqrt{3} \end{pmatrix} , \qquad \mu_3 = \begin{pmatrix} 0 \\ -1/\sqrt{3} \end{pmatrix} , \tag{5.5.8}$$

  These weights are shown on the left side of figure 13. Notice that the difference of any two weights is a root, in agreement with lemma 5.22.

  This figure describes the arrangement of the three quarks u, d and s in a figure where the two axes are isospin *and* strangeness*, two properties of fundamental particles you will learn.*

- *We now consider the antifundamental representation. From the point of view of the group* SU(3)*, this differs from the fundamental in that we use $\bar{U}$ rather than U to transform our vectors. In other words, $\rho_{\text{anti}}(U) \equiv \bar{U}$. To see what this means at the level of the Lie algebra, recall $U = e^u$, where u is antihermitian. So $\bar{U} = e^{\bar{u}} = e^{-u^t}$. So, at the level of the Lie algebra* su(3)*, this means $\rho_{\text{anti}}(u) = -u^t$. (**Exercise**: check that this is a representation.) Hence, to determine the weights, we can just transpose the H_a in (5.2.12) (which doesn't affect them, since they are diagonal) and multiply them by a minus. We conclude, then, that the weights of the anti-fundamental are just minus the ones for the fundamental. This is shown in figure 13, on the right. This figure now describes the arrangement of the three* anti*quarks $\bar{u}$, $\bar{d}$ and $\bar{s}$.*

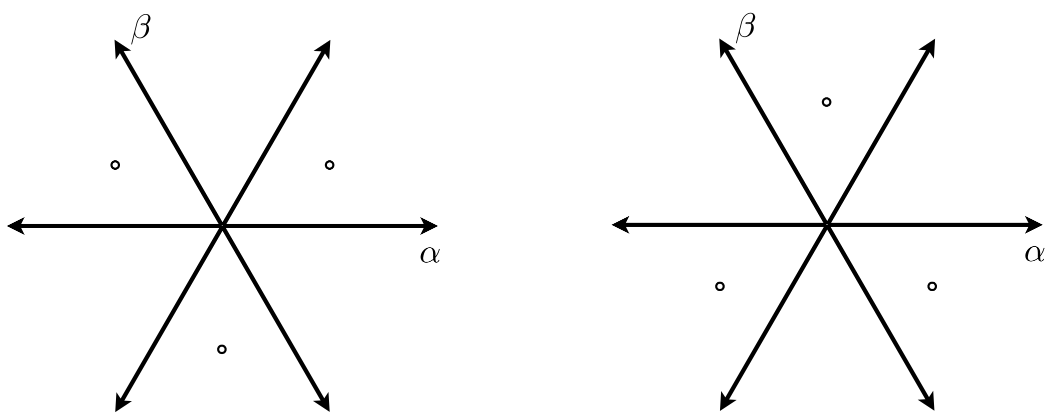


Figure 13: The dots denote the weights for the fundamental (left) and anti-fundamental (right), and root system of $\mathrm{sl}(3, \mathbb{C}) \cong \mathrm{su}(3)_{\mathbb{C}}$ (compare diagram in figure 11).

- *The weight of the adjoint representation are the roots, as pointed out in 5.10. This figure describes the arrangement of eight mesons, again in a plot where the two axes are isospin and strangeness. Mesons are particles made of a quark and an antiquark. Group theoretically, we know that the adjoint* 8 *is contained in the tensor product* $3 \otimes \overline{3}$ *of a fundamental and of an antifundamental representation (which are the representations we have associated to quarks and to antiquarks, respectively).*

  We can compute the roots of $3 \otimes \overline{3}$ by using lemma 5.23. If we take all the possible sums between the weights of the fundamental and the weights of the antifundamental, we obtain nine possible vectors: the six non-zero roots $\pm\alpha$, $\pm\beta$, $\pm(\alpha+\beta)$, and three copies of the null vector. Since su(3) has rank 2, we know that $\mathfrak{h} = L_{\underline{0}}$ has dimension

2. So we can arrange the non-zero roots and two copies of the null vector to make up the adjoint. This leaves us with one copy of the zero root, which corresponds to the singlet. So we find again that $3 \otimes \overline{3} = 8 \oplus 1$.

As you noticed, I couldn't resist the temptation of pointing out that some notable elementary and composite particles arrange themselves just like weights of representations of SU(3). In a nutshell, the reason is that the Lagrangian that describes quarks and antiquarks has an approximate SU(3) symmetry (called "flavor" symmetry) that rotates them. As you may have heard, there are actually six different quarks, and this SU(3) can be extended to include them; but such an extension is not as good an approximation, and it is seldom used. I'll leave you the pleasure of learning more about the physical implications of group theory in your elementary particles classes. For now, let us go back to the study of the mathematics of Lie algebras.

Recall that we have defined (see def. 5.7) a set of positive roots Φ_+, and one of negative roots Φ_-.

Definition 5.11. *A weight of a representation is called* maximal *if it is annihilated by the action of all the positive roots:* $x_\alpha v_\mu = 0$, $\forall \alpha \in \Phi_+$.

Example 5.13. *Let us find maximal weights for the* su(3) *representations in example 5.12. We have to look for weights which are annihilated by the positive roots* α, β, $\alpha+\beta$. *For the fundamental (see figure 13, left), we see that the upper-right vertex of the "triangle" is maximal; in terms of (5.5.8), this is* μ_1. *The other two vertices are not annihilated by all positive roots. For the antifundamental (see figure 13, right), we see that the upper vertex of the "triangle" is maximal; in terms of (5.5.8), this is* $-\mu_3$. *For the adjoint representation, weights are the same as roots:* $\alpha+\beta$ *is then a maximal weight.*

For the representations in the example, there is only one maximal weight. This is true in general:

Theorem 5.24. *An irreducible representation* V *has only one maximal weight. Moreover, it is possible to reconstruct the whole representation from it.*

Proof. Given a maximal weight μ, consider the space X_μ generated by V_μ and all the subspaces

$$x_{-\alpha_1} \dots x_{-\alpha_n} V_\mu = \Pi_{a=1}^n x_{-\alpha_a} V_\mu \tag{5.5.9}$$

obtained acting with any number of simple roots α_i on V_μ. In other words:

$$X_\mu = \oplus_n \Pi_{a=1}^n x_{-\alpha_a} V_\mu \ , \tag{5.5.10}$$

where α_a are simple roots. We will now see that this vector space X_μ is invariant under the action of $\mathfrak{g}$. First of all, lemma 5.22 tells us that X_μ is invariant under the action of the Cartan subalgebra $\mathfrak{h}$. So we have to check that the x_α also keep it invariant.

Recall from theorem 5.19) that any positive root can be written as $\sum_a x_a \alpha_a$, with α_a simple and a_a non-negative integers. Because of this, it is enough to check that X_μ is left invariant by *simple* roots (and by their opposites). For example, if a positive root is sum of two simple roots, $\alpha = \alpha_1 + \alpha_2$, we have $x_\alpha = [x_{\alpha_1}, x_{\alpha_2}]$, and if x_{α_1} and x_{α_2} leave X_μ invariant, so does x_α. Similarly, for negative roots, it is enough to check that X_μ is left invariant by the $-\alpha_a$.

Actually, it is obvious that $x_{-\alpha_a}$ leaves X_μ invariant, from its definition X_μ. The action of x_{α_a} is more complicated: one has to use $x_{\alpha_a} x_{-\alpha_b} = [x_{\alpha_a}, x_{-\alpha_b}] + x_{-\alpha_b} x_{\alpha_a}$ several times, so as to bring the "annihilator" x_{α_a} to the left of the "creators" $x_{-\alpha_b}$. The commutator $[x_{\alpha_a}, x_{-\alpha_b}]$ is zero, except when $a = b$ (since $\alpha_a - \alpha_b$ is never a root: see lemma 5.17). When $a = b$, the commutator is in $\mathfrak{h}$; so it gives again an element of X_μ. At the end of this process, x_{α_a} will act directly on V_μ, which gives zero, by definition of maximal weight. Summing up, we get that x_{α_a} leaves X_μ invariant.

Now, X_μ is left invariany by the action of $\mathfrak{g}$; in other words, it is a representation of $\mathfrak{g}$. A priori, $X_\mu \subset V$; but since V is irreducible, $X_\mu = V$. So we have reconstructed V from the maximal weight μ. This proves the second statement.

Let us now prove that there is only one maximal weight. From (5.5.9) and lemma 5.22 we see that any weight for the representation X_μ can be written as

$$\mu - \sum_a a_a \alpha_a \ , \tag{5.5.11}$$

where a_a are positive integers, and α_a are simple roots. Suppose now that there is another maximal weight μ'. If we write it as (5.5.11), we see that the action of the positive root $\sum_a a_a \alpha_a$ does not annihilate it. This contradicts the hypothesis that μ' is maximal. This proves that the maximal weight is unique. □

Given any vector $\mu \in \mathbb{C}^r$, we can define a representation X_μ of $\mathfrak{g}$ by (5.5.10); by construction, μ will be the maximal weight of X_μ. In general, however, this representation will be infinite-dimensional! Notice indeed that in (5.5.10) we have put no bound on the number of "creators" $x_{-\alpha_a}$ we can apply to V_μ. We would now like to understand if X_μ can be finite-dimensional, and, if so, under what conditions.

We will first find a property which is common to all weights of a finite-dimensional representation (maximal or not).

Recall that for any root α one can define a subalgebra $s_\alpha \subset \mathfrak{g}$ isomorphic to $\mathrm{sl}(2,\mathbb{C})$. In section 5.2 many results were proven by considering the action of these s_α subalgebras on other roots β. This suggests that it might also be useful to consider the action of s_α on the weights μ of a representation. From the definition of H_α in (5.2.20), we see that

$$H_\alpha v_\mu = \frac{\alpha \cdot \mu}{\alpha^2} v_\mu \ . \tag{5.5.12}$$

This formula generalizes (5.2.24). Recall that $H_\alpha v_\mu$ stands for $\rho(H_\alpha)v_\mu$: we have decided to drop the symbols ρ in this subsection to make our formulas more readable. The "ad" in (5.2.24) is nothing but ρ for the adjoint representation.

Lemma 5.25. *If α is a root and μ is a weight of a finite-dimensional representation,*

$$2\frac{\alpha \cdot \mu}{\alpha^2} \in \mathbb{Z} \ . \tag{5.5.13}$$

Proof. Recall that, for a representation of $\mathrm{sl}(2,\mathbb{C})$, the weight m (which is in that case the eigenvalue of ad_H) is a half-integer for any finite-dimensional representation. Hence (5.5.12) implies (5.5.13). □

Theorem 5.26. *A vector $\mu \in \mathbb{C}^r$ is the maximal weight for a finite-dimensional representation X_μ if and only if*

$$D_a \equiv 2\frac{\alpha_a \cdot \mu}{\alpha_a^2} \tag{5.5.14}$$

is a non-negative integer for every simple root α_a. (Such a μ is called a dominant weight*.)*

Proof. We will first show that (5.5.14) is necessary. We know already that (5.5.14) should be an integer, thanks to lemma 5.25. Moreover, if α is a positive root and μ is a maximal weight, by definition $\alpha + \mu = 0$; hence, $x_\alpha v_\mu = 0$. Hence v_μ has maximal weight for s_α; but we know that $\mathrm{sl}(2,\mathbb{C})$ should be positive.

Showing that (5.5.14) is sufficient is more complicated; we are only going to sketch the proof. The idea is that the summands (5.5.9) in (5.5.10) will vanish when there are "too many creators" $x_{-\alpha_a}$. In other words, one of the (5.5.9) will be annihilated by any further action of the $x_{-\alpha_a}$. This will happen when the weight μ' of such a subspace is related to the maximal weight μ by a Weyl reflection w_α. Indeed, $\mu' = w_\alpha(\mu)$ is annihilated by the roots $w_\alpha(\alpha_a)$. □

To make the idea clearer, we will now see an example of how the proof of sufficiency really works.

Example 5.14. • *For* $sl(2,\mathbb{C})$*, the highest weight state is* $|l\rangle$*; the states of the form (5.5.9) are nothing but the* $Y^k|l\rangle \equiv |l-k\rangle$*. If the representation is finite-dimensional, these states vanish for some high enough* k*: this happens when we reach the state* $|-l\rangle$*, which is obtained from* $|l\rangle$ *by the Weyl reflection that exchanges the two roots* ± 1*.*

- *Let us now consider the representations of* $sl(3,\mathbb{C})$ *that we saw in example 5.12. In all these cases, all weights can be reached from the maximal weight by acting with the negative roots. Their action however vanishes after a few steps: it ends in some weights which are obtained from the maximal weight by a Weyl reflection. This manifests itself in the fact that the weight systems for each of these representations is invariant under some subgroup of the Weyl group.*

Definition 5.12. *The integers* D_a *in (5.5.14) are called* Dynkin indices*. A weight is said to be* fundamental *if all* D_a *are zero except one.*

So we have classified the dominant weights (i. e. the weights that can be maximal weights for a representation) in terms of the Dynkin indices. This means we have classified all possible irreducible representations.

Example 5.15. *We can compute easily the Dynkin indices of the representations in example 5.12. The simple roots are* $\alpha = \alpha_1$ *and* $\beta = \alpha_3$ *in (5.2.14).*

For the fundamental representation, we saw in example 5.13 that the maximal weight is μ_1 *(see (5.5.8)). Using (5.5.14) we get* $D_1 = 2\frac{\alpha.\mu_1}{\alpha^2} = 1$ *and* $D_2 = 2\frac{\beta.\mu_1}{\beta^2} = 0$*: so the indices are* $D_a = (1,0)$*. So* μ_1 *is a fundamental weight.*

For the antifundamental representation, we saw in example 5.13 that the maximal weight is $-\mu_3$ *(see (5.5.8)). Using (5.5.14) we get* $D_a = (0,1)$*. So* $-\mu_3$ *is the other fundamental weight.*

For the adjoint representation, we saw in example 5.13 that the maximal weight is $\alpha+\beta$*: so we get* $D_a = (1,1)$*.*

In figure 14 we show all the dominant weights for $\mathrm{sl}(3,\mathbb{C})$.

For $\mathrm{SU}(N)$, there is a general rule to relate Dynkin indices to the Young diagrams we saw in section 4.9. Recall that a Young diagram for $\mathrm{SU}(N)$ is at most $N-1$ boxes deep; in other words, diagrams have $N-1$ rows of boxes. Call r_i the number of boxes on the i-th row, starting from the top of the diagram. Now the Dynkin indices of this representation are given by $D_a = r_a - r_{a-1}$.

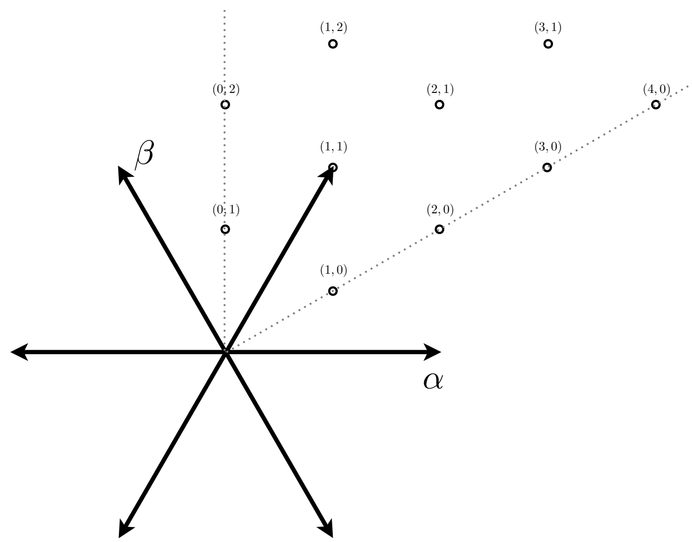


Figure 14: The lattice of dominant weights for $\mathrm{sl}(3,\mathbb{C}) = \mathrm{su}(3)_{\mathbb{C}}$. The tiny numbers (D_1, D_2) are the Dynkin indices.

Example 5.16. *The Dynkin indices of the following representations of* $\mathrm{SU}(3)$*:*

$$\ydiagram{1}\ , \qquad \ydiagram{1,1}\ , \qquad \ydiagram{2,1}\ , \qquad \ydiagram{4,1} \tag{5.5.15}$$

are respectively $(1,0)$, $(0,1)$, $(1,1)$, $(3,1)$. *The first three representations are the fundamental, the antifundamental and the adjoint, respectively.*

There are many applications of this formalism. We won't have time to see them all, but here is a particularly useful one:

Theorem 5.27. ***(Weyl dimension formula.)*** *The dimension of the representation* ρ_μ *with maximal weight* μ *is*

$$\dim(\rho_\mu) = \Pi_{\alpha\in\Phi_+} \frac{(\mu+\delta)\cdot\alpha}{\delta\cdot\alpha}\ , \tag{5.5.16}$$

where $\delta \equiv \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$ *is the sum of all positive roots.*

Example 5.17.

- *For su(2), we have* $\delta = 1/2$*, since 1 is the only positive root. The maximal weight* μ *of a representation is what we usually call* l*. So we have* $\dim(\rho_l) = \frac{(l+1/2)\cdot 1}{1/2\cdot 1} = 2l+1$.
- *For su(3), we have* $\delta = 1/2(\alpha+\beta+\alpha+\beta) = \alpha+\beta$*. The two fundamental weights* μ_1 *and* $-\mu_3$*, in the notation of (5.5.8); these are the ones labeled* $(1,0)$

and $(0,1)$*, respectively, in 14. Decompose the maximal weight* μ *as a sum of these two:* $\mu = n_1\mu_1 + n_2(-\mu_3)$*. To compute each factor in (5.5.16), it is convenient to reduce both numerator and denominators to objects of the form (5.5.13), which we know how to compute. For example,* $\frac{(\mu+\alpha+\beta)\cdot\alpha}{(\alpha+\beta)\cdot\alpha} = \frac{(\mu+\alpha+\beta)\cdot\alpha/\alpha\cdot\alpha}{(\alpha+\beta)\cdot\alpha/\alpha\cdot\alpha} = \frac{n_1/2+1/2}{1/2} = 2n_1+1$*. Computing the other factors in a similar fashion, we get*

$$\dim(\rho_{(n_1,n_2)}) = \frac{1}{2}(n_1+1)(n_2+1)(n_1+n_2+2)\ . \tag{5.5.17}$$

Another possible application of the formalism is a generalization of the "total angular momentum" operator $\ell_i\ell_i$ for su(2). The peculiarity of this operator is that it commutes with all the generators in every representation. (I have to specify that we are using a representation, because the product of two generators doesn't really make sense in a Lie algebra. The proper way to talk about such operators would be to introduce something called the "universal enveloping algebra", which is an algebra where such products make sense.). It is easy to generalize this: introduce a basis e_i for $\mathfrak{g}$; let k_{ij} be the Killing form in this basis, and k^{ij} its inverse. Then we can write the "Casimir operator"

$$C \equiv k^{ij} e_i e_j \tag{5.5.18}$$

which commutes with all generators: $[C, e_k]$. This implies that C should be proportional to the identity in any irreducible representation. One can show that the proportionality constant is given by:

$$C = \mu\cdot(\mu+2\delta)1\ . \tag{5.5.19}$$

Example 5.18. *For su(2), we have* $C = \ell_i\ell_i$*. The proportionality constant in (5.5.19) is* $\mu\cdot(\mu+2\delta) = l(l+1)$*, which is what we know from our QM classes.*

Actually, C in (5.5.18) is not the only product of generators that commutes with every element of $\mathfrak{g}$. There are in fact r such operators. They can all be obtained by expanding the expression

$$\det_{ij}(e_k f^i{}_{kj} - \lambda\delta^i{}_j) \tag{5.5.20}$$

in powers of λ.

To summarize, we have classified all finite-dimensional representations of semisimple Lie algebras:

- There is a one-to-one correspondence between irreducible representations and dominant weights (the μ such that the D_a defined in (5.5.14) are positive integers).
- The dominant weights are arranged in a lattice, generated by the fundamental weights.

Exercises

Many exercises have been already given in the main text, but here are a few more.

1. Find all subgroups of $\mathbb{Z}_k$. Are they normal? Is $\mathbb{Z}_k$ simple for any k?
2. Find all the groups (up to isomorphism) with four elements.
3. How many homomorphisms are there from $\mathbb{Z}_k$ to $\mathbb{Z}$? and from $\mathbb{Z}$ to $\mathbb{Z}_k$?
4. Find the conjugacy classes of the dihedral group D_k, for any k.
5. Find all the representations of the dihedral group D_4. Check formula (3.2.13) in this case.
6. Show that the rotation group of the cube (namely, the subgroup of SO(3) whose elements send a cube to itself) is given by S_4. (Hint: consider the four diagonals of the cube as the objects permuted by S_4.)
7. Consider the group D_3, and its representations ρ^0 (the trivial representation) and ρ^2 given in (2.2.7). Check equation (3.2.8) for $(\rho^0_{11}, \rho^2_{12})$ and $(\rho^2_{12}, \rho^2_{12})$.
8. Using the abstract definition with generators and structure constants, find all Lie algebras of dimension 2. Find at least one representation for every one of them. Find the corresponding Lie groups.
9. Consider the space t_k of upper-triangular matrices. Show that it is a Lie algebra. What is the corresponding Lie group? For $k = 2$ and $k = 3$, find the structure constants of t_k (in whatever basis you want).
10. The *center* $Z(G)$ of a group G is the set of elements $z \in G$ that commute with all G: $Z = \{zg = gz,\ \forall g \in G\}$. Show that $Z(G)$ is a group. Find $Z(\mathrm{SU}(N))$.
11. The center is a normal subgroup, $Z(G) \triangleleft G$. (Right?) So for example $\mathrm{SU}(2)/Z(\mathrm{SU}(2))$ is a group. It also has another name: which one?
12. Find an injective homomorphism from $\mathrm{u}(N)$ to $\mathrm{so}(2N)$. (Hint: think of $\mathbb{C}^N$ as $\mathbb{R}^{2N}$).
13. Show that $\mathrm{U}(N) = \mathrm{Gl}(N, \mathbb{C}) \cap \mathrm{O}(2N)$.
14. Find some elements in $\mathrm{Sl}(2, \mathbb{R})$ that are not exponentials of elements in $\mathrm{sl}(2, \mathbb{R})$.
15. What is $\mathrm{su}(N)_{\mathbb{C}}$?

16. Define the *Riemann sphere* S to be the union of the complex plane $\mathbb{C}$ with a single "point at infinity" ∞. (S is topologically S^2, and you can think of $\mathbb{C}$ as the plane $\mathbb{R}^2$ in the stereographic projection in figure 5.) A *Möbius transformation* is a holomorphic map $f : S \to S$ defined by

$$f(z) = \frac{az+b}{cz+d} \,. \tag{5.5.21}$$

Show that the set of all Möbius transformations is a group. Show that it is isomorphic to $\mathrm{Gl}(2,\mathbb{C})/\mathbb{C}^* \equiv \mathrm{PGl}(2,\mathbb{C})$.

17. What is the subgroup of the group of Möbius transformations (see previous exercise) that sends the upper-half plane into itself?

18. We saw in example 4.31 that the representation of SU(2) of spin $l \in \frac{1}{2}\mathbb{Z}$ can be seen as the space of completely symmetric tensors with $k = 2l$ indices. List all such tensors for fixed k, and check that there are $k+1 = 2l+1$ of them (which should be the dimension of the representation of spin l).

19. Write the Young diagram corresponding to the adjoint representation for su(N). (For $N = 3$, the answer is given in example 4.32.)

20. In SU(3), decompose in irreducible representations the tensor products $3 \otimes 6$ and $3 \otimes 3 \otimes 3$.

21. Find the $\mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C})$ labels (l_+, l_-) for symmetric tensors $g_{\mu\nu}$.

22. Write the representations $(1,0)$ and $(0,1)$ of the Lorentz group in terms of tensors with vector indices μ alone (i. e. without any spinorial indices).

23. Given the spin representation $(1/2, 0)$ and the vector representation $(1/2, 1/2)$, decompose their tensor product in irreducible representations of the Lorentz group. What kind of "tensors" corresponds to these irreducible representations?

24. Find the structure constants of the Lie algebra of the Lie group E(2) (the Euclidean group in two dimensions; see (2.3.16)). Is it solvable, semisimple, or neither?

25. Compute the roots of $\mathrm{sl}(3,\mathbb{C}) = \mathrm{su}(3)_{\mathbb{C}}$ for a choice of $\{H_a\}$ different from the one we took in example 5.3 (for example, $H_1 = \mathrm{diag}(1,-1,0)$ and $H_2 = \mathrm{diag}(0,1,-1)$). Compute k_{ab}. Notice that both the roots and k_{ab} have different expressions from the ones we found in example 5.3. Check, however, that with definition (5.2.11) it is still true that all roots have the same length, and that the angles between them are still multiples of $\pi/3$ (as in (5.2.16)).

26. Draw the weights of the SU(3) representation with Dynkin indices $(3, 0)$. Check lemma 5.23 for $3 \otimes 3 \otimes 3$. [The resulting diagrams shows all the possible particles made up of three quarks (also called *baryons*) of the type u, d and s, in a plot where isospin and strangeness are on the two axes. Each of the three 3 you are tensoring represents a quark.]

27. We have seen in figure 14 the lattice of dominant weights for su(3). Draw the lattices of dominant weights for the other three semisimple Lie algebras or rank 2. (Recall that we gave the corresponding root systems in figure 11.)

28. Using figure 11 and no computation, identify a subalgebra of g_2 isomorphic to su(3). Decompose now the adjoint of g_2 in representations of this subalgebra (similarly to the way we have decomposed the adjoint in representations of the subalgebras $s_\alpha \cong \mathrm{sl}(2, \mathbb{C})$, for example in figure 10).

29. Consider a subalgebra $\mathrm{su}(3) \subset \mathrm{su}(4)$ defined by $s \in \mathrm{su}(3)$, $\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{su}(4)$. Decompose the adjoint of su(4) in representations of su(3). In the spirit of the previous exercise, use this information to guess the three-dimensional root system of su(4).

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